A.N. SHARKOVSKY
INSTITUTE OF MATHEMATICS
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UKRAINE

Sharkovsky Ordering

The **Sharkovsky ordering** describes the coexistence of cycles with different periods for discrete-time dynamical systems given by maps $f : I \to I$ where I is an interval in the real line **R** and, possibly, $I = \mathbf{R}$. One can also say that it provides a forcing relation for the existence of cycles of certain periods due to the presence of a cycle of another period.

The Sharkovsky ordering is the following ordering of natural numbers

$$1 \prec 2 \prec 2^2 \prec 2^3 \prec \cdots \prec 2^n \prec \ldots \cdots \prec 7 \cdot 2^n \prec 5 \cdot 2^n \prec 3 \cdot 2^n \prec \cdots \prec 7 \cdot 2 \prec 5 \cdot 2 \prec 3 \cdot 2 \prec \cdots \prec 9 \prec 7 \prec 5 \prec 3 .$$

Let f^n, $n \geq 1$, denote the n-th iteration of f, i.e., $f^n = f(f^{n-1})$, where f^0 is the identity map. The point $x \in I$ is a periodic point of period m ($m \geq 1$) for f, if $f^m(x) = x$ and $f^n(x) \neq x$ for any $1 \leq n < m$. In this case, the points $x, f(x), \ldots, f^{m-1}(x)$ form a periodic orbit or a cycle of period m.

Theorem (Sharkovsky, 1964) *If a continuous map of an interval into itself has a cycle of period m, then it has a cycle of any period $\tilde{m} \prec m$. Moreover, for any m there exists a continuous map that has a cycle of period m but does not have cycles of periods $\overline{m}$, $m \prec \overline{m}$.*

The lecture will deal with some properties of the ordering, its possible generalizations on various classes of maps, spaces, and the history of this ordering birth.

In 2019, the book ”Sharkovsky ordering” (authors A.M.Blokh (USA), M.Yu.Matviichuk (Canada), and myself) should appear in Springer. I will also talk about the content of this book.

AMIT CHATTOPADHYAY
INTERNATIONAL INSTITUTE OF INFORMATION TECHNOLOGY
BANGALORE
INDIA

1. Simplicial Homology: Computing Betti Numbers

Simplicial homology is a classical tool for computing Betti numbers in higher-dimensional data. This introductory talk will present a technique for computing the homology groups and an incremental algorithm for computing the Betti numbers. We will also see the relation between the Euler characteristic and the Betti numbers.

2. Morse Theory

Morse theory plays an important role in topological analysis of scalar data. This introductory talk will present some basic concepts of Morse theory, Morse inequalities and their relation with the homology theory. We will also see an alternative method for computing the Euler characteristic using the number of critical points with different indices corresponding to a Morse function.

ANDREI TETENOV
GORNO-ALTAISK STATE UNIVERSITY
RUSSIA

Self-similar dendrites and their main properties

To understand the nature of self-similar dendrites, we begin with an overview of self-similar Jordan curves. We begin with explanation of self-similar zippers and multizippers as a natural way to construct self-similar curves and discuss the properties of their attractors. Then we show, that if a self-similar Jordan arc is not a straight line segment, then it can be represented by some multizipper. Main ideas of this approach are applied to the study of post-critically finite self-similar dendrites. We discuss self-similar dendrites generated by polygonal systems and formulate their basic propeties. We show that the notion of a main tree gives a key to understanding of a wider class of self-similar dendrites. Finally we define special kind of finite trees called m-sprouts and show that topological classes of self-similar dendrites may be defined by inverse limits of topological spaces associated with n-th powers of m-sprouts.

ANIMA NAGAR
INDIAN INSTITUTE OF TECHNOLOGY
DELHI
INDIA

1. Transitivity and related topics

We consider the topological dynamical systems $T \times X \to X$, given as $(t,x) \to tx$, on a topological space X with T as a acting group or semigroup. In this talk, we discuss various aspects of topological transitivity for the systems (X,T).

This talk is dedicated to the memory of Sergiy Kolyada.

2. Sensitivity and related topics

A topological dynamical system (X,f) is sensitive or sensitively depends on initial condition, if there is a positive constant δ such that in each non-empty open subset there are distinct points whose iterates will be δ–apart at same instance. This dynamical property, though being a very weak one, brings in the essence of unpredictability in the system. In this talk we survey some properties implied by and implying sensitivity.

CHRISTOPHER GOOD
UNIVERSITY OF BIRMINGHAM
UNITED KINGDOM

Shadowing

Let $f : X \to X$ be a function on the compact metric space X. A sequence $(x_i)_{i\in\omega}$ is a δ-pseudo-orbit if $d(f(x_i),x_{i+1}) < \delta$. The point x is said to ϵ-shadow (x_i) if $d(f^i(x),x_i) < \epsilon$. Pseudo-orbits arise naturally in the numerical calculation of orbits. It turns out that pseudo-orbits can often be tracked within a specified tolerance by real orbits, in which case f is said to have the shadowing, or pseudo-orbit tracing, property; we say that f has shadowing if for every $\epsilon > 0$, there is a $\delta > 0$ such that every δpseudo-orbit is ϵ-shadowed by some point x.

Clearly this is of importance when trying to model a system numerically, especially when the system is expanding and errors might grow exponentially. However, shadowing is also of theoretical importance and the notion can be traced back to the analysis of Anosov and Axiom A diffeomorphisms. Sinai isolated subsystems of Anosov diffeomorphisms with shadowing and Bowen proved explicitly that for the larger class of Axiom A diffeomorphisms, the shadowing property holds on the nonwandering set. However, Bowen had already used shadowing implicitly as a key step in his proof that the nonwandering set of an Axiom A diffeomorphism is a factor of a shift of finite type. The notion of structural stability of a dynamical system was instrumental in the definitions of both Anosov and Axiom A diffeomorphisms and Pilyugin and others show that shadowing plays a key role in stability theory. Shadowing is also key to characterizing omega-limit sets. Moreover, Walters showed that a shift space has shadowing if and only if it is of finite type.

In these two workshop talks we will take a whistle stop tour of shadowing and some of its variants, ending with some structural results of the author and Jonathan Meddaugh.

DAN BURGHELEA
OHIO STATE UNIVERSITY
USA

New Topological invariants inspired from data analysis and dynamics

(i). "large data" viewed as a large collection of points in a large dimensional euclidean space (point cloud data) and its geometrization as a shape (most frequent simplicial complex) equipped with a single or multivalued map.

(ii). The new computable topological invariants (bar codes, Jordan cells) introduced via homology for such pair (shape and map) and their relevance for data. These invariants are referred to as "topological persistence" invariants and are named "bar codes" and "Jordan cells"and can be collected as configuration of points in plane.

(iii). Simple examples and what one can derive out of these new invariants.

(iv). I will provide one of the (many possible)rigorous definition of the barcodes and Jordan cells which make them computer friendly, and I will discuss their relevance for dynamics of a flow on a nice metric space.

(v). I will describe the basic properties of these invariants.

(vi).I will show how to recover the main conclusions of Morse Theory out of these invariants and implicitly obtain an alternative approach to Morse and Morse-Novikov theory which works in a considerably larger generality.

DOMINIC KWIETNIAK
JAGIELLONIAN UNIVERSITY IN KRAKOW
POLAND

On properties of dynamical systems induced on hyperspaces and probability measures

Every dynamical system given by a continuous map f from a compact metric space X to itself induces in a natural way two systems: one defined on the hyperspace of X, that is, the space of all nonempty compact subsets of X and one defined on the space of all Borel probability measures with supports contained in X. The hyperspace system is given by the map taking a nonempty closed subset of X to its image through f, and the second system is defined by the push forward map. We will discuss the connections between the dynamical properties of the original systems and induced systems.

KAROLY SIMON
BUDAPEST UNIVERSITY OF TECHNOLOGY AND ECONOMICS
HUNGARY

A gentle introduction to the dimension theory of some non-Markovian repellers

The two talks I present at the workshop serve as an introduction to my plenary talk on the conference which is about our very recent result (joint with Balázs Bárány and Michał Rams) related to the dimension theory of some non-Markovian repellers on the plane.

Michael Barnsley introduced a family of some fractals sets which are repellers of piecewise affine systems. The study of these fractals was motivated by some problems that arose in fractal image compression but the results we obtained can be applied for the computation of the Hausdorff dimension of the graph of some functions, like generalized Takagi functions and fractal interpolation functions.

First I introduce these fractal sets then I collect and discuss the tools both from one-dimensional dynamics and from fractal geometry that we need to study the dimension theory of these fractal sets.

More precisely, the topological pressure is defined for continuous transformation and continuous functions on compact metric space. Franz Hofbauer introduced a notion of topological pressure (Hofbauer pressure) for some families of piecewise continuous and piecewise expanding maps and piecewise continuous potentials. The Hofbauer pressure is the most important tool to study the repellers mentioned above. So, I give a gentle introduction of the Hofbauer pressure. In particular we learn about a Theorem of Franz Hofbauer and Marius Urbanski related to the existence of conformal measures.

On the other hand, I describe some of the most important very recent results from the dimension theory of self-affine sets which are relevant from the point of the study of the Barnsley-repellers mentioned above.

HENK BRUIN
UNIVERSITY OF VIENNA
AUSTRIA

Interval dynamics and Inverse limit spaces

The inverse limit space of a non-invertible dynamical system is the space of all its backward orbits, equipped with product topology. If the dynamical system is a unimodal map (e.g. tent map) on the interval, then the inverse limit space, when embedded into the plane, has similarities with certain skew-product and also with Lozi attractors (although Loziattractors are even more complicated). In this mini-course I want to explain how to describe unimodal inverse limits (UILs), how dynamical properties of the unimodal map translate into topological properties of the UIL and vice versa. Also the classification problem known as the Ingram conjecture (tent-maps with different slopes have non-homeomorphic UILs) will be discussed. This is partially based on former joint results with Karen Brucks, Sonja Stimac, Ana Anuˇsi´c, JernejˇCinˇc.ˇ

HISAO KATO
UNIVERSITY OF TSUKUBA
JAPAN

Chaotic continua in chaotic dynamical systems

During the last thirty years or so, many interesting connections between dynamical systems and continuum theory have been studied by many mathematicians. Many complicated spaces frequently appear in chaotic dynamical systems. Such spaces play important roles in order to investigate behaviors of the dynamics. We are interested in the following fact that chaotic topological dynamics should imply the existence of complicated topological structures of underlying spaces. In many cases, such spaces are indecomposable continua. We know that many indecomposable continua often appear as chaotic attractors of dynamical systems. Also, in many cases, the composants of such indecomposable continua are strongly related to stable or unstable (connected) sets of the dynamics. For instance, in continuum theory and the theory of dynamical systems, the Knaster continuum (= Smale's horse shoe), the pseudo-arc, solenoids and Wada's lakes (= Plykin attractors) etc., are well-known as such indecomposable continua. The theory of indecomposable continua is one of the most interesting branches of continuum theory in topology.

By use of ergodic theory method, Blanchard, Glasner, Kolyada and Maass proved that if a map $f : X \to X$ of a compact metric space X has positive topological entropy, then there is an uncountable δ-scrambled subset of X for some $\delta > 0$ and hence the dynamics (X,f) is Li-Yorke chaotic. Huang and Ye studied local entropy theory and they gave a characterization of positive topological entropy by use of entropy tuples. Kerr and Li developed local entropy theory and gave a new proof of the theorem of Blanchard, Glasner, Kolyada and Maass. Moreover, they proved that X contains a Cantor set Z which yields more chaotic behaviors. Barge and Diamond showed that for piecewise monotone surjections of graphs, the conditions of having positive topological entropy, containing a horse shoe and the inverse limit space containing an indecomposable subcontinuum are all equivalent. Mouron proved that if X is an arc-like continuum which admits a homeomorphism with positive topological entropy, then X contains an indecomposable subcontinuum. As an extension of the Mouron's theorem, Darji and Kato proved that if X is a G-like continuum for a graph G and X admits a homeomorphism f with positive topological entropy, then X contains an indecomposable subcontinuum. Moreover, if the graph G is a tree, then there is a pair of two distinct points x and y of X such that the pair (x,y) is an IE-pair of f and the irreducible continuum between x and y in X is an indecomposable subcontinuum.

In this talk, for any graph G we define a new notion of "free tracing property by free G-chains" on G-like continua and we prove that a positive topological entropy homeomorphism f of a G-like continuum X admits a Cantor set Z in X such that any sequence $(z_1,z_2,...,z_n)$ of points in Z is an IE-tuple of f and Z has the free tracing property by free G-chains. Our main theorem is a dynamical and geometric structure theorem of positive topological entropy homeomorphism of Glike continua. Also, we show that the similar result can be obtained for positive topological entropy "monotone" maps. Also, we give characterization theorems of continua containing indecomposable subcontinua.

JAMES YORKE
UNIVERSITY OF MARYLAND
USA

1.Generalized Lorenz equations on a three-sphere

Edward Lorenz published several models with chaos during his career. I will explore how these models are related, which will lead us to a differential equation on the 3-sphere, in the spirit of Lorenz. This concerns a paper with that title: Eur. Phys. J. Special Topics 226, 1751–1764 (2017).

2.What's the point: Finding fixed points

Topological proofs of the existence of fixed points appear to be completely non constructive. I will show how they can be turned into effective numerical methods for finding the fixed points.

JAVIER CAMARGO
UNIVERSIDAD INDUSTRIAL DE SANTANDER
COLOMBIA

Hyperspaces of continua

We present the notion of hyperspace. We show some geometric models. In addition to present some general properties, we present some problems in the theory of hyperspaces of continua.

JOSEPH AUSLANDER
UNIVERSITY OF MARYLAND
USA

1.Proximality. regional proximality, and the Veech relation

We review the basic properties of the proximal and regionally proximal relations (P and RP), their connection with the distal and equicontinuous structure relations, and conditions under which they are equivalence relations. The Veech relation (V) is a subset of RP, but surprisingly frequently coincides with it in minimal flows. We also consider the relation of V with almost automorphy and higher order RP.

2.The Furstenberg Structure Theorem

Hillel Furstenberg's theorem on the structure of distal minimal flows is more than fifty years old, but it still dominates topological dynamics. We will indicate the main steps in the proof, note some consequences, and discuss related theorems in topological dynamics and ergodic theory.

KIT CHAN
BOWLING GREENSTATE UNIVERSITY
USA

1. Some Basic Properties of Hypercyclic Operators

Using a few classical examples and the invariant subspace problem, we motivate the definition of a hypercyclic operator on a Banach space. We state a sufficient condition for an uncountable family of operators to have a dense G_δ set of common hypercyclic vectors. Then we exhibit a few examples of such uncountable families. Finally we switch our focus to some results on extending of an operator defined on a Hilbert subspace to a hypercyclic operator on the whole Hilbert space.

2.Some Examples of Hypercyclic Operators and Universal Sequences of Operators

Many examples of hypercyclicity take place in analytic function spaces, such as spaces of entire functions, Hardy spaces, Bergman spaces, and Dirichlet spaces. Using unique features of these analytic function spaces, we explore properties of some hypercyclic operators, such as spectral properties, orbital properties, as well as their hypercyclicity with respect to different topologies of the spaces.

KRZYSZTOF LESNIAK
NICOLAUS COPERNICUS UNIVERSITY
POLAND

Iterated function systems: a topological approach

1) Invariant sets ▪ The Knaster–Tarski principle ▪ Multifunctions and IFSs; the Hutchinson operator; invariance aka generalized selfsimilarity ▪ Measures of noncompactness and condensing maps ▪ Darbo–Sadovskii's joint generalization of the Schauder and Banach fixed point principles ▪ The Birkhoff theorem on a minimal set; maximal invariant sets

2) Attractors ▪ Global maximal attractors in IFSs per analogiam PDEs' maximal attractors ▪ The Hutchinson attractor ▪ The Barnsley□Vince strict attractor; pointwise strict attractor ▪ Coding map and contractivity ▪ Probabilistic chaos game algorithm ▪ Derandomized chaos game algorithm for contractive IFSs.

PATRIZIO FROSINI
UNIVERSITY OF BOLOGNA
ITALY

1.The natural pseudo-distance in topological data analysis

In this talk we will present the concept of natural pseudo-distance associated with a given group G of homeomorphisms, and the link of this pseudo-metric with persistent homology. Some results concerning the natural pseudo-distance will be also illustrated.

2.The use of group equivariant non-expansive operators in topological data analysis

In this talk we will describe how the theory of group equivariant non-expansive operators (GENEOs) can cast new light on the use of persistent homology in topological data analysis. We will also illustrate some results concerning the topological and geometrical structure of the space of GENEOs.

ROBERT DEVANEY
BOSTON UNIVERSITY
USA

1.The Fractal Geometry of the Mandelbrot Set

In this lecture we describe several folk theorems concerning the Mandelbrot set. While this set is extremely complicated from a geometric point of view, we will show that, as long as you know how to add and how to count, you can understand this geometry completely. We will encounter many famous mathematical objects in the Mandelbrot set, like the Farey tree and the Fibonacci sequence. And we will find many soon-to-be-famous objects as well, like the "Devaney" sequence. There might even be a joke or two in the talk.

2.Cantor and Sierpinski, Julia and Fatou: Crazy Topology in Complex Dynamics

In this talk, we shall describe some of the rich topological structures that arise as Julia sets of certain complex functions including the exponential and rational maps. These objects include Cantor bouquets, indecomposable continua, and Sierpinski curves.

ROMAN HRIC
MATEJ BEL UNIVERSITY
SLOVAKIA

An Introduction to Topologically Minimal Systems

Topological minimality is one of the most fundamental properties of dynamical systems. Basically it says that a system with this property cannot be reduced or decomposed to smaller, more elementary subsystems, at least, in the point of view of topological dynamics.

We say that a dynamical system is *topologically minimal* if all the orbits are dense (in accordance with common practice in topological dynamics, we will omit the adverb "topologically"). During the lectures we will see also equivalent definitions which make it clear why we call these systems minimal. We will be concerned mostly with discrete-time and continuoustime systems represented by maps, homeomorphisms, and (semi)flows but we will mention also a more general setting of (semi)group actions on topological spaces which shall give us a perspective to better understand some notions. We will also see that, in fact, we can consider non-equivalent natural notions of minimality depending on what exactly we mean by orbits.

After principal definitions and introductory examples we will proceed to fundamental results on existence and properties of minimal systems. For example, one of basic questions is whether every dynamical system, even if it is not minimal itself, has a minimal subsystem (such subsystems are also called *minimal sets*). Another essential problem, in principle impossible to solve in full generality, is what can be said about topological properties of minimal systems or minimal sets, especially, what kind of spaces admit a minimal system. An important topic is also connection between minimality and recurrence. To elucidate theoretical concepts, examples ranging from symbolic systems to surface flows will be exploited.

It has to be understood that existing theory on minimal systems forms nowadays an immense body of results, examples, etc., thus it is impossible to present a comprehensive introduction into the topic in such a short time and, inevitably, some issues legitimately considered as essential to the topic have to be neglected.

SABER ELAYDI
TRINITY UNIVERSITY
USA

Theory of Triangular maps, Theory of monotone maps

In these talks, we will investigate the global dynamics of triangular maps, monotone maps and a class of non-monotone maps.

Applications to population dynamics will be presented. The use of skew-product construction will be utilized to study non-autonomous systems.

VARADACHARIAR KANNAN
UNIVERSITY OF HYDERABAD
INDIA

Chaos for the maps on the real line and the interval

The following four theorems will be proved.

Every topologically transitive map is Devaney-chaotic.

Every D-chaotic map has a 6-cycle.

Every map with a 3-cycle admits a scrambled set. (Li-Yorke chaotic).

Every chaotic map on R has an unbounded set of critical points.

VIJAY NATARAJAN
INDIAN INSTITUTE OF SCIENCE
INDIA

Reeb Graph: Computation and Applications

Scientific data is often represented as a scalar function defined on a geometric domain. For example, X-ray crystallographers compute the electron density at various points of a molecular crystal using diffraction measurements from x-rays bouncing on the crystal. It is essential to know the electron density to perform structure related studies of the molecule. The electron density is a scalar function defined on a subset of the three-dimensional Euclidean space. Another example is Magnetic Resonance Imaging (MRI), a popular technique used to take pictures of different slices of the human body. The atom density over the slice is mapped to a gray-scale image and studied by radiologists to detect tumors. The scalar function in this case is the density defined on a set of two-dimensional planes stacked together and can be viewed as a function defined on a subset of 3D Euclidean space.

Scientific data is becoming increasing large in size and feature-rich. This makes it difficult to analyze and not amenable to direct visualization and exploration. One approach towards solving this problem is to automatically extract features from the dataset, construct an abstract representation, and present them to the user. Computing the topological properties of the scalar function is a step in this direction and has been studied within the field of computational topology. The preimage of a real value is called a level set of the scalar function. The level set may consist of multiple connected components. We are interested in the evolution of the topology of the level set as the function value increases. The Reeb graph expresses the evolution of connected components of level sets as a graph. Nodes of the graph correspond to critical points of the function.

In the first talk, I will define the Reeb graph, list its properties, and describe an algorithm to construct the graph for a scalar function defined on a simply connected domain. In this case, the graph does not have any cycles. In the second talk, I will sketch an efficient algorithm for computing the Reeb graph of a generic function. This algorithm employs a split-and-compute approach and leverages the simple algorithm for simply connected domains. I will also describe various applications of the Reeb graph in scientific visualization, graphics, and geometry processing.

WLODZIMIERZ J CHARATONIK
MISSOURI UNIVERSITY OF SCIENCE AND TECHNOLOGY
USA

1.Self homeomorphic spaces

We will recall the definitions of various kinds of self homeomorphic spaces: self homeomorphic, strongly self homeomorphic, pointwise self homeomorphic, and strongly pointwise self homeomorphic. We also discuss properties of attractors of iterated function systems as self homeomorphic spaces. The talk will be illustrated by many examples.

2.Multivalued inverse limits

In this workshop I would like to familiarize the listeners with the notion of inverse limits with multivalued bonding functions. The notion was first defined my William T. Mahavier in 2004 and immediately get very popular among investigators in continuum theory. W. T. Ingram wrote two books on the subject and many talks in topology conferences are devoted to those inverse limits. I hope to give elementary examples, so no specific knowledge is necessary to understand the talk.