Module 6
Columns and Struts
Columns and Struts

• Any member subjected to axial compressive load is called a column or Strut.

• A vertical member subjected to axial compressive load – COLUMN (Eg: Pillars of a building)

• An inclined member subjected to axial compressive load - STRUT

• A strut may also be a horizontal member

• Load carrying capacity of a compression member depends not only on its cross sectional area, but also on its length and the manner in which the ends of a column are held.
• Equilibrium of a column – Stable, Unstable, Neutral.

• Critical or Crippling or Buckling load – Load at which buckling starts

• Column is said to have developed an elastic instability.
Classification of Columns

- According to nature of failure – short, medium and long columns

  1. **Short column** – whose length is so related to its c/s area that *failure occurs mainly due to direct compressive stress* only and the role of bending stress is negligible

  2. **Medium Column** - whose length is so related to its c/s area that *failure occurs by a combination of direct compressive stress and bending stress*

  3. **Long Column** - whose length is so related to its c/s area that *failure occurs mainly due to bending stress* and the role of direct compressive stress is negligible
Euler’s Theory

• Columns and struts which fail by buckling may be analyzed by Euler’s theory

• Assumptions made
  
  • the column is initially straight
  • the cross-section is uniform throughout
  • the line of thrust coincides exactly with the axis of the column
  • the material is homogeneous and isotropic
  • the shortening of column due to axial compression is negligible.
Case (i) Both Ends Hinged

\[ EI \frac{d^2 y}{dx^2} = M = -Py \]
\[ EI \frac{d^2 y}{dx^2} = M = -Py \]

The equation can be written as \[ \frac{d^2 y}{dx^2} + \alpha^2 y = 0 \] where \[ \alpha^2 = \frac{P}{EI} \]

The solution is \[ y = A \sin \alpha x + B \cos \alpha x \]

At \( x = 0, y = 0, \therefore B = 0 \)

at \( x = l, y = 0 \) and thus \( A \sin \alpha l = 0 \)

If \( A = 0, y \) is zero for all values of load and there is no bending.

\[ \therefore \sin \alpha l = 0 \text{ or } \alpha l = \pi \] (considering the least value)

or \( \alpha = \frac{\pi}{l} \)

\[ \therefore \text{ Euler crippling load, } P_e = \alpha^2 EI = \frac{\pi^2 EI}{l^2} \]
Case (ii) One end fixed other free

\[ EI \frac{d^2 y}{dx^2} = M = P(a - y) = Pa - Py \]
\[ EI \frac{d^2 y}{dx^2} = M = P(a - y) = Pa - Py \]

\[ \frac{d^2 y}{dx^2} + \alpha^2 y = \frac{P \cdot a}{EI} \quad \text{where} \quad \alpha^2 = \frac{P}{EI} \]

The solution is \( y = A \sin \alpha x + B \cos \alpha x + \frac{P \cdot a}{EI \alpha^2} \)

\[ = A \sin \alpha x + B \cos \alpha x + a \]

\( x = 0, \ y = 0, \ \therefore B = -a; \)

\[ x = 0, \ \frac{dy}{dx} = 0 \]

or \( A \alpha \cos \alpha x - B \alpha \sin \alpha x = 0 \)

or \( A = 0 \)

\( y = -a \cos \alpha x + a = a(1 - \cos \alpha x) \)
At $x = l$, $y = a$, \[ a = a(1 - \cos \alpha l) \]

or $\cos \alpha l = 0$ or $\alpha l = \frac{\pi}{2}$ (considering the least value)

$\alpha = \frac{\pi}{2l}$

\[ \therefore \text{Euler crippling load, } P_e = \alpha^2 EI = \frac{\pi^2 EI}{4l^2} \]
Case (iii) Fixed at both ends

\[ EI \frac{d^2 y}{dx^2} = -Py + M \]
\[ EI \frac{d^2 y}{dx^2} = -Py + M \]

\[ \frac{d^2 y}{dx^2} + \alpha^2 y = \frac{M}{EI} \quad \text{where} \quad \alpha^2 = \frac{P}{EI} \]

The solution is  
\[ y = A \sin \alpha x + B \cos \alpha x + \frac{M}{EI\alpha^2} = A \sin \alpha x + B \cos \alpha x + \frac{M}{P} \]

\[ x = 0, y = 0, \therefore B = -\frac{M}{P}; \]

\[ x = 0, \quad \frac{dy}{dx} = 0 \]

or  
\[ A\alpha \cos \alpha x - B\alpha \sin \alpha x = 0 \quad \text{or} \quad A = 0 \]

\[ \therefore y = -\frac{M}{P} \cos \alpha x + \frac{M}{P} = \frac{M}{P}(1 - \cos \alpha x) \]
At \( x = l, y = 0 \), \( 0 = \frac{M}{P}(1 - \cos \alpha l) \) or \( \cos \alpha l = 1 \).

or \( \alpha l = 2\pi \) (considering the least value)

or \( \alpha = \frac{2\pi}{l} \)

\[ \therefore \text{Euler crippling load, } P_e = \alpha^2 EI = \frac{4\pi^2 EI}{l^2} \]
Case (iv) One end fixed, other hinged

\[ EI \frac{d^2 y}{dx^2} = -Py + R(l - x) \]
\[ EI \frac{d^2 y}{dx^2} = -Py + R(l-x) \]

\[ \frac{d^2 y}{dx^2} + \alpha^2 y = \frac{R(l-x)}{EI} \quad \text{where} \quad \alpha^2 = \frac{P}{EI} \]

The solution is
\[ y = A \sin \alpha x + B \cos \alpha x + \frac{R(l-x)}{EI\alpha^2} \]

\[ = A \sin \alpha x + B \cos \alpha x + \frac{R}{P}(l-x) \]

At \( x = 0, y = 0 \), \( \therefore B = -\frac{Rl}{P} \);

At \( x = 0, \frac{dy}{dx} = 0 \)

or \( A \alpha \cos \alpha x - B \alpha \sin \alpha x - \frac{R}{P} = 0 \)

For \( A \), \( A = \frac{R}{P\alpha} \)
\[ y = \frac{R}{P\alpha} \sin \alpha x - \frac{Rl}{P} \cos \alpha x + \frac{R}{P} (l - x) \]

At \( x = l, y = 0 \), \[ 0 = \frac{R}{P\alpha} \sin \alpha l - \frac{Rl}{P} \cos \alpha l \]

or \( \tan \alpha l = \alpha l \)

\[ \alpha l = 4.49 \text{ rad} \quad \text{(considering the least value)} \]
\[ \alpha = \frac{4.49}{l} \]

\[ \therefore \text{Euler crippling load, } P_e = \alpha^2 EI = \frac{4.49^2 EI}{l^2} = \frac{20.2EI}{l^2} \approx \frac{2\pi^2 EI}{l^2} \]
Equivalent Length \( (l_e) \)

Euler’s load can be expressed as

\[
P_e = \frac{\pi^2 EI}{l_e^2}
\]

where \( l_e^2 \) is referred as *equivalent length* of the column which takes into account the type of fixing of the ends.
The equivalent lengths for different types of end conditions are

(i) both ends hinged, \( l_e = l \)
(ii) one end fixed and the other free, \( l_e = 2l \)
(iii) both ends fixed, \( l_e = l/2 \)
(iv) one end fixed, other hinged, \( l_e = l/\sqrt{2} \)
Limitations of Euler’s Formula

• Assumption – Struts are initially perfectly straight and the load is exactly axial.
• There is always some eccentricity and initial curvature present.
• In practice a strut suffers a deflection before the Crippling load.
• Critical stress ($\sigma_c$) – average stress over the cross section

$$\sigma_c = \frac{P_e}{A} = \frac{\pi^2 EI}{Al_e^2} = \frac{\pi^2 EAk^2}{Al_e^2}$$

$$\sigma_c = \frac{\pi^2 E}{(l_e/k)^2}$$

• $l/k$ is known as **Slenderness Ratio**
Slenderness Ratio

• **Slenderness ratio** is the ratio of the length of a column and the radius of gyration of its cross section

• Slenderness Ratio = \( \frac{l}{k} \)

The Radius of Gyration \( k_x \) of an Area \( A \) about an axis \( x \) is defined as:

\[
I_x = k_x^2 A \\
k_x = \sqrt{\frac{I_x}{A}}
\]
Rankine’s Formula
OR
Rankine-Gorden Formula

• Euler’s formula is applicable to long columns only for which $l/k$ ratio is larger than a particular value.
• Also doesn’t take in to account the direct compressive stress.
• Thus for columns of medium length it doesn’t provide accurate results.
• Rankine forwarded an empirical relation
\[
\frac{1}{P} = \frac{1}{P_c} + \frac{1}{P_e}
\]

where \( P = \text{Rankine's crippling load} \)
\( P_c = \text{ultimate load for a strut} = \sigma_u \cdot A \), constant for a material
\( P_e = \text{Eulerian load for a strut} = \pi^2 \frac{EI}{l^2} \)

- For short columns, \( P_e \) is very large and therefore \( 1/P_e \) is small in comparison to \( 1/P_c \). Thus the crippling load \( P \) is practically equal to \( P_c \).
- For long columns, \( P_e \) is very small and therefore \( 1/P_e \) is quite large in comparison to \( 1/P_c \). Thus the crippling load \( P \) is practically equal to \( P_e \).
\[
\frac{1}{P} = \frac{1}{P_e} + \frac{1}{P_c} = \frac{P_e + P_c}{P_c P_e} = \frac{P_c}{1 + P_c / P_e} = \frac{\sigma_c \cdot A}{1 + \frac{\sigma_c A \cdot l^2}{\pi^2 EI}} = \frac{\sigma_c \cdot A}{1 + \frac{\sigma_c A \cdot l^2}{\pi^2 EA k^2}}
\]

\[P = \frac{\sigma_c \cdot A}{1 + a \left( \frac{l}{k} \right)^2}\]

where \(\sigma_c\) is the crushing stress

\(a\) is the Rankine's constant \((\sigma_c / \pi^2 E)\)
A Factor of Safety may be considered for the value of $\sigma_c$ in the above formula.

\[
\frac{1}{P} = \frac{1}{P_c} + \frac{1}{P_e}
\]

\[
\frac{1}{P} = \frac{P_e + P_c}{P_c P_e}
\]

\[
P = \frac{P_c P_e}{P_e + P_c} = \frac{P_c}{1 + P_c / P_e} = \frac{\sigma_c \cdot A}{1 + \frac{\sigma_c A \cdot l^2}{\pi^2 EI}}
\]

\[
P = \frac{\sigma_c \cdot A}{1 + \frac{\sigma_c A \cdot l^2}{\pi^2 EA k^2}}
\]

where $\sigma_c$ is the crushing stress

$a$ is the Rankine's constant ($\sigma_c / \pi^2 E$)

- A Factor of Safety may be considered for the value of $\sigma_c$ in the above formula
• Rankine’s formula for columns with other end conditions

\[ P = \frac{\sigma_c \cdot A}{1 + a \left( \frac{l_e}{k} \right)^2} \]