Department of Applied Electronics & Instrumentation

COURSE HANDOUT
THIRD SEMESTER
RSET VISION

To evolve into a premier technological and research institution, moulding eminent professionals with creative minds, innovative ideas and sound practical skill, and to shape a future where technology works for the enrichment of mankind.

RSET MISSION

To impart state-of-the-art knowledge to individuals in various technological disciplines and to inculcate in them a high degree of social consciousness and human values, thereby enabling them to face the challenges of life with courage and conviction.

DEPARTMENT VISION

To evolve into a centre of academic excellence, developing professionals in the field of electronics and instrumentation to excel in academia and industry.

DEPARTMENT MISSION

Facilitate comprehensive knowledge transfer with latest theoretical and practical concepts, developing good relationship with industrial, academic and research institutions thereby moulding competent professionals with social commitment.
PROGRAM EDUCATIONAL OBJECTIVES

PEOI: Graduates will possess engineering skills, sound knowledge and professional attitude, in electronics and instrumentation to become competent engineers.

PEOII: Graduates will have confidence to design and develop instrument systems and to take up engineering challenges.

PEOIII: Graduates will possess commendable leadership qualities, will maintain the attitude to learn new things and will be capable to adapt themselves to industrial scenario.

PROGRAM OUTCOMES

Engineering Graduates will be able to:

PO1. Engineering knowledge: Apply the knowledge of mathematics, science, engineering fundamentals, and an engineering specialization to the solution of complex engineering problems.

PO2. Problem analysis: Identify, formulate, review research literature, and analyze complex engineering problems reaching substantiated conclusions using first principles of mathematics, natural sciences, and engineering sciences.

PO3. Design/development of solutions: Design solutions for complex engineering problems and design system components or processes that meet the specified needs with appropriate consideration for the public health and safety, and the cultural, societal, and environmental considerations.
PO4. Conduct investigations of complex problems: Use research-based knowledge and research methods including design of experiments, analysis and interpretation of data, and synthesis of the information to provide valid conclusions.

PO5. Modern tool usage: Create, select, and apply appropriate techniques, resources, and modern engineering and IT tools including prediction and modeling to complex engineering activities with an understanding of the limitations.

PO6. The engineer and society: Apply reasoning informed by the contextual knowledge to assess societal, health, safety, legal and cultural issues and the consequent responsibilities relevant to the professional engineering practice.

PO7. Environment and sustainability: Understand the impact of the professional engineering solutions in societal and environmental contexts, and demonstrate the knowledge of, and need for, sustainable development.

PO8. Ethics: Apply ethical principles and commit to professional ethics and responsibilities and norms of the engineering practice.

PO9. Individual and team work: Function effectively as an individual, and as a member or leader in diverse teams, and in multidisciplinary settings.

PO10. Communication: Communicate effectively on complex engineering activities with the engineering community and with society at large, such as, being able to comprehend and write effective reports and design documentation, make effective presentations, and give and receive clear instructions.
PO11. Project management and finance: Demonstrate knowledge and understanding of the engineering and management principles and apply these to one’s own work, as a member and leader in a team, to manage projects and in multidisciplinary environments.

PO12. Life-long learning: Recognize the need for, and have the preparation and ability to engage in independent and life-long learning in the broadest context of technological change.
Students of the program
PSO 1: will have sound technical skills in electronics and instrumentation.
PSO 2: will be capable of developing instrument systems and methods complying with standards.
PSO 3: will be able to learn new concepts, exhibit leadership qualities and adapt to changing industrial scenarios
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**TOTAL** | **26/30** | **22/26** |
MAT 201 PARTIAL DIFFERENTIAL EQUATION AND COMPLEX ANALYSIS
### COURSE INFORMATION SHEET

<table>
<thead>
<tr>
<th>PROGRAMME: COMMON EXCEPT CS/IT</th>
<th>DEGREE: BTECH</th>
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<tbody>
<tr>
<td>PROGRAMME: AE/CE/EC/EEE/ME</td>
<td>DEGREE: B. TECH</td>
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<tr>
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<td>UNIVERSITY: A P J ABDUL KALAM TECHNOLOGICAL UNIVERSITY</td>
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**COURSE: PARTIAL DIFFERENTIAL EQUATIONS AND COMPLEX ANALYSIS**

- **SEMMESTER:** III
- **CREDITS:** 4

**COURSE CODE:** MAT 201

**REGULATION:** UG

**COURSE TYPE:** CORE

**COURSE AREA/DOMAIN:** ENGINEERING MATHEMATICS

**CONTACT HOURS:** 3+1 (Tutorial) hours/Week.

### SYLLABUS:

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<tr>
<th>UNIT</th>
<th>DETAILS</th>
<th>HOURS</th>
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<tbody>
<tr>
<td>I</td>
<td>PARTIAL DIFFERENTIAL EQUATIONS</td>
<td>8</td>
</tr>
<tr>
<td>II</td>
<td>APPLICATION OF PARTIAL DIFFERENTIAL EQUATIONS</td>
<td>10</td>
</tr>
<tr>
<td></td>
<td>One dimensional wave equation- vibrations of a stretched string, derivation, solution of the wave equation using method of separation of variables, D'Alembert's solution of the wave equation, One dimensional heat equation, derivation, solution of the heat equation</td>
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<td>III</td>
<td>COMPLEX VARIABLE-DIFFERENTIATION</td>
<td>9</td>
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<td></td>
<td>Complex function, limit, continuity, derivative, analytic functions, Cauchy-Riemann equations, harmonic functions, finding harmonic conjugate, Conformal mappings w=z^2, w=e^z. Linear fractional transformation w=1/z. fixed points, Transformation w=z sin z (From sections 17.1, 17.2 and 17.4 only mappings w=z^2, w=e^z, w=^-1, w=sinz and problems based on these transformation need to be discussed.</td>
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<td>IV</td>
<td>COMPLEX VARIABLE-INTEGRATION</td>
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<td></td>
<td>Complex integration, Line integrals in the complex plane, Basic properties, First evaluation method-indefinite integration and substitution of limit, second evaluation method-use of a representation of path, Contour integrals, Cauchy integral theorem (without proof) on simply connected domain, Cauchy integral</td>
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</tbody>
</table>
theorem (without proof) on multiply connected domain Cauchy Integral formula (without proof), Cauchy Integral formula for derivatives of a analytic function, Taylor's series and Maclaurin series.

**V COMPLEX VARIABLE-RESIDUE INTEGRATION**
Laurent’s series (without proof), zeros of analytic functions, singularities, poles, removable singularities, essential singularities, Residues, Cauchy Residue theorem (without proof), Evaluation of definite integral using residue theorem. Residue integration of real integrals and rational functions. Improper integrals of the form \( \int_{-\infty}^{\infty} f(x) \, dx \).

**TOTAL HOURS** 45

**TEXT/REFERENCE BOOKS:**

<table>
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<tr>
<th>T/R</th>
<th>BOOK TITLE/AUTHORS/PUBLICATION</th>
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**COURSE PRE-REQUISITES:**

<table>
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<tr>
<th>C.CODE</th>
<th>COURSE NAME</th>
<th>DESCRIPTION</th>
<th>SEM</th>
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<tbody>
<tr>
<td></td>
<td>A basic course in partial differentiation and complex numbers</td>
<td>To develop basic ideas on partial differentiation and Complex numbers etc.</td>
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</table>

**COURSE OBJECTIVES:**

1. To equip the students with methods of solving partial diff. equation with first order
2. To familiarize them with the concept of boundary value problems which have many applications in engineering like heat and wave equations
3. To understand the basic theory of functions of a complex variable, calculus of complex valued functions and conformal transformations
# COURSE OUTCOMES:

<table>
<thead>
<tr>
<th>SNO</th>
<th>DESCRIPTION</th>
<th>Bloom's Taxonomy Level</th>
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<tbody>
<tr>
<td><strong>CO 1</strong></td>
<td><strong>Identify the</strong> concept and the solution of partial differential equation.</td>
<td>Remember (Level 1)</td>
</tr>
<tr>
<td><strong>CO 2</strong></td>
<td><strong>Analyze</strong> and solve one dimensional wave equation and heat equation.</td>
<td>Analyse (Level 4)</td>
</tr>
<tr>
<td><strong>CO 3</strong></td>
<td><strong>Understand</strong> complex functions, its continuity differentiability with the use of Cauchy- Riemann equations.</td>
<td>Understand (Level 2)</td>
</tr>
<tr>
<td><strong>CO 4</strong></td>
<td><strong>Evaluate</strong> complex integrals using Cauchy's integral theorem and Cauchy's integral formula, understand the series expansion of analytic function</td>
<td>Evaluate (Level 5)</td>
</tr>
<tr>
<td><strong>CO 5</strong></td>
<td>Understand the series expansion of complex function about a singularity and <strong>apply</strong> residue theorem to compute several kinds of real integrals.</td>
<td>Apply (Level 3)</td>
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## CO-PO AND CO-PSO MAPPING

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## JUSTIFICATIONS FOR CO-PO MAPPING

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<tbody>
<tr>
<td><strong>CO 1-PO 1</strong></td>
<td>3</td>
<td>Fundamental knowledge in PDE will help to analyse the Engineering problems very easily</td>
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<tr>
<td><strong>CO 1-PO 2</strong></td>
<td>3</td>
<td>Basic knowledge for the solution of PDE will help to model various problems in engineering fields</td>
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<tr>
<td><strong>CO 1-PO 3</strong></td>
<td>3</td>
<td>Solution of PDE will help to simplify problems with high complexity in Engineering</td>
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<tr>
<td>CO 1-PO 4</td>
<td>3</td>
<td>Non-linear partial differential equations will help to design solutions to various complex engineering problems</td>
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<td>2</td>
<td>Find the difference between complete integral and singular integral of a partial differential equation</td>
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<td>CO 1-PO 6</td>
<td>1</td>
<td>Variable separable form will help to enrich the analysis of engineering problem</td>
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<tr>
<td>CO 1-PO 10</td>
<td>2</td>
<td>Analyse the method of separation of variables for solving PDE</td>
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<tr>
<td>CO 1-PO 12</td>
<td>2</td>
<td>Methods for the solutions of PDE will give a thorough knowledge in the application problem</td>
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<tr>
<td>CO 2-PO 1</td>
<td>3</td>
<td>Will able to analyse various methods of solutions of boundary value problems</td>
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<tr>
<td>CO 2-PO 2</td>
<td>3</td>
<td>Will able to analyse various methods of solutions of initial value problems</td>
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<tr>
<td>CO 2-PO 3</td>
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<td>Analyse one dimensional wave equation</td>
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<tr>
<td>CO 2-PO 4</td>
<td>3</td>
<td>Analyse one dimensional heat equation</td>
</tr>
<tr>
<td>CO 2-PO 5</td>
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<td>Analyse D-Alembert's solution of wave equation</td>
</tr>
<tr>
<td>CO 2-PO 6</td>
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<td>Analyse Fourier solution of heat equation</td>
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<td>CO 2-PO 10</td>
<td>2</td>
<td>Apply the concept of above in boundary application</td>
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<tr>
<td>CO 2-PO 12</td>
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<td>Apply the concept in the solution of heat equation</td>
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<tr>
<td>CO 3-PO 1</td>
<td>3</td>
<td>Understand the idea of complex variable and functions</td>
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<td>Understand the idea of continuity of complex valued functions</td>
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<td>CO 3-PO 3</td>
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<td>Understand the idea of differentiability of complex valued function</td>
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<tr>
<td>CO 3-PO 4</td>
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<td>Understand the concept of Differentiability and Cauchy Riemann equations</td>
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<td>Understand the engineering application of analytic function in fluid mechanics</td>
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<td>CO 3-PO 6</td>
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<td>Understand the idea about stream and potential function</td>
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<td>Understand the idea about harmonic function</td>
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<td>Understand the idea about harmonic conjugate</td>
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<td>Evaluation Cauchy's integral theorem</td>
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<td>Evaluation of complex integration</td>
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<td>Evaluation of Cauchy's integral formula</td>
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<tr>
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<td>3</td>
<td>Evaluation of complex integral using Cauchy's integral formula</td>
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<td>CO 4-PO 5</td>
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<td>Understanding of the idea of complex integration</td>
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<tr>
<td>CO 4-PO 6</td>
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<td>Understanding of idea about multi connected region</td>
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<tr>
<td>CO 4-PO 10</td>
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<td>Series expansion of analytic function</td>
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Understand the significance of series expansion in practical problems

Knowledge about the singularities

Understanding of residues and its evaluation

Apply the residue theorem for evaluation of real integrals

Apply the residue theorem for evaluation of integrals

Derivation of residue theorem

Apply the application of residue theorem

Apply the residue theorem for evaluation of improper integrals

Apply the residue theorem for evaluation of trigonometric functions

**JUSTIFICATIONS FOR CO-PSO MAPPING**

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<tr>
<td>CO1-PSO2</td>
<td>Medium</td>
<td>Students will be get the capability of developing instrument systems and methods complying with standards by identifying the concept and the solution of partial differential equation</td>
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<tr>
<td>CO2-PSO3</td>
<td>Medium</td>
<td>Students will be able to learn new concepts and adapt to changing industrial scenario by analysing and solving one dimensional wave equation and heat equation.</td>
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<tr>
<td>CO3-PSO1</td>
<td>High</td>
<td>Students will have sound technical skills in electronics and instrumentation by understanding complex functions, its continuity differentiability with the use of Cauchy- Riemann equations.</td>
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<tr>
<td>CO5-PSO3</td>
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<td>Students will be able to learn new concepts and exhibit leadership qualities by understanding the series expansion of complex function about a singularity and applying residue theorem to compute several kinds of real integrals</td>
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**GAPS IN THE SYLLABUS - TO MEET INDUSTRY/PROFESSIONAL REQUIREMENTS:**

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PROPOSED ACTIONS: TOPICS BEYOND SYLLABUS/ASSIGNMENT/INDUSTRY VISIT/GUEST LECTURER/NPTEL ETC
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WEB SOURCE REFERENCES / ICT ENABLED TEACHING LEARNING RESOURCES:

2. https://www.youtube.com/watch?v=Fh8m6ZdFaqU
3. https://www.youtube.com/watch?v=GmlcbqdvIgc
4. https://www.youtube.com/watch?v=2ZBcbFhrfOg
5. https://www.youtube.com/watch?v=o77UV7YrWvw
6. https://www.youtube.com/watch?v=jd_t8jUJJfA
7. https://www.youtube.com/watch?v=2l4jKiGy238
8. https://www.youtube.com/watch?v=uliv9TzeD6o

DELIVERY/INSTRUCTIONAL METHODOLOGIES:

- ☑ CHALK & TALK
- ☑ STUD. ASSIGNMENT
- ☑ WEB RESOURCES
- ☑ LCD/SMART BOARDS
- ☐ STUD. SEMINARS
- ☐ ADD-ON COURSES

ASSESSMENT METHODOLOGIES-DIRECT

- ☑ ASSIGNMENTS
  - ☐ STUD. SEMINARS
  - ☑ TESTS/MODEL EXAMS
  - ☑ UNIV. EXAMINATION
- ☐ STUD. LAB PRACTICES
  - ☐ STUD. VIVA
  - ☐ MINI/MAJOR PROJECTS
  - ☐ CERTIFICATIONS
- ☐ ADD-ON COURSES
  - ☐ OTHERS

ASSESSMENT METHODOLOGIES-INDIRECT

- ☑ ASSESSMENT OF COURSE OUTCOMES (BY FEEDBACK, ONCE)
  - ☑ STUDENT FEEDBACK ON FACULTY (TWICE)
- ☐ ASSESSMENT OF MINI/MAJOR PROJECTS BY EXT. EXPERTS
  - ☐ OTHERS

Prepared by

Approved by
## COURSE PLAN

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MAT 201

RAJAGIRI SCHOOL OF ENGINEERING & TECHNOLOGY
DEPARTMENT OF MATHEMATICS
TUTORIAL / ASSIGNMENT/ UNIT-WISE QUESTION BANK RECORD BOOK
COURSE:- MAT 201; PARTIAL DIFFERENTIAL EQUATIONS AND COMPLEX ANALYSIS
Branch: COMMON FOR ALL BRANCHES (EXCEPT CS & IT)

MODULE 1 PARTIAL DIFFERENTIAL EQUATIONS

TUTORIAL

1. Form the differential equation satisfied by \( z = ax + by + a^2 + b^2 \)
2. Find the differential equation of all spheres of fixed radius whose centres lie in the xy - plane.
3. Form the partial differential equation from \( z = f(x^2 - y^2) \).
4. Form the partial differential equation from \( z = x^2 f(y) + y^2 g(x) \).
5. Form the differential equation satisfied by \( F(xy + z^2, x + y + z) = 0. \)
6. Solve the PDE \( \frac{\partial^2 z}{\partial x \partial y} = \frac{x}{y} + a \) by direct integration
7. Solve the PDE \( \frac{\partial^2 z}{\partial x^2} = a^2 z \) given that when \( z = 0, \frac{\partial z}{\partial x} = a \sin y \) and \( \frac{\partial z}{\partial y} = 0 \) by direct integration
8. Solve the PDE \( y^2 zy + x^2 zy = y^2 z \)
9. Solve the PDE \( pqz + qzx = xy \)
10. Solve the PDE \( xp - yq = y^2 - x^2 \)
11. Solve the PDE \( (z - y)p + (x - z)q = y - x \)
12. Solve the PDE \( zpq + p + q \) by Charpit's method
13. Using Charpit's method solve \( (x^2 + y^2)z = pz \)
14. Solve by using Charpit's method \( z = pqzxy \)
15. Using the method of separation of variables solve \( 4 \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 3u, \ u(0, y) = 3e^{-y} - e^{-5y} \)

ASSIGNMENT

1. Form the differential equation satisfied by \( (a - x)^2 + (y - b)^2 + z^2 = c^2 \)
2. Find the differential equation of all spheres whose centres lie on the z-axis.
3. Form the differential equation satisfied by \( z = f(x^2 + y^2) + x + y \)
4. Form the differential equation satisfied by \( z = yf(x) + zg(y) \).
5. Form the differential equation satisfied by \( F(x + y + z, x^2 + y^2 + z^2) = 0. \)
6. Solve the PDE \( \frac{\partial^2 z}{\partial x \partial t} = e^{-t} \cos z \) by direct integration

7. Solve the PDE \( \frac{\partial^2 z}{\partial y^2} = z \), given that when \( y = 0, z = e^x \) and \( \frac{\partial z}{\partial y} = e^{-x} \) by direct integration

8. Solve the PDE \( p - q = \log (x + y) \)

9. Solve the PDE \( p + 3q = 5z + \tan(y - 3x) \)

10. Solve the PDE \( 2p + 3q = 1 \)

11. Solve the PDE \( x(x^2 - x^3)p + y(x^2 - x^3)q = z(y^2 - x^2) \)

12. Solve the PDE \( q = 3p^2 \) by Charpit’s method

13. Using Charpit’s method solve \( px + qy = pq \)

14. Using Charpit’s method solve \( p^2 + q^2 - 2px - 2qy + 1 = 0 \)

15. Using the method of separation of variables solve \( \frac{\partial u}{\partial x} + 2 \frac{\partial u}{\partial y} = 0 \), \( u(x, 0) = 4e^{-x} \)

**UNIT-WISE QUESTION BANK**

1. Form the differential equation satisfied by \( xyz = \phi(x + y + z) \).

2. Form the differential equation satisfied by \( z = f(x) + e^y g(x) \).

3. Form the differential equation satisfied by \( z = f_1(y + 2x) + f_2(y - 3x) \).

4. Solve the PDE \( \frac{\partial^2 z}{\partial x \partial y} = z^2 y \) by direct integration

5. Solve the PDE \( xp + yq = 3z \)

6. Solve the PDE \( p\sqrt{x} + q\sqrt{y} = z \)

7. Solve the PDE \( (y^2 + z^2 - x^2)p - 2xyq + 2xz = 0 \)

8. Solve the PDE \( z(x + y)p + z(x - y)q = x^2 + y^2 \) (Hint: \( x, -y, z \) and \( y, x, -z \) are the multipliers)

9. Solve the PDE \( q = px + p^2 \) by Charpit’s method

10. Using Charpit’s method solve \( p^2 x + q^2 y = z \)

11. Using Charpit’s method solve \( px + pq + qy = yz \)

12. Using Charpit’s method solve \( 1 + p^2 = qz \)

13. Using the method of separation of variables solve \( \frac{\partial u}{\partial x} = 4 \frac{\partial u}{\partial y} \) where \( u(0, y) = 8e^{-3y} \)

14. Using the method of separation of variables solve \( u(x, y) = 1 + e^{-3y} \), when \( x = 0 \) for \( x = 0 \) for all values of \( y \)

15. Using the method of separation of variables solve \( \frac{\partial^2 u}{\partial x \partial t} = e^{-t} \cos x \) where \( u = 0 \) when \( \frac{\partial u}{\partial t} = 0 \), and \( u = 0 \) when \( t = 0 \)
Course Handout

MAT 201: Partial Differential Equations and Complex Analysis

Module 1: Partial Differential Equations

Bina R, Eva Cherian

Department of Mathematics
Rajagiri School of Engineering and Technology

Pre requisites

(i) Calculating partial derivatives
(ii) Evaluating partial derivatives at a point
(iii) Chain rule for partial derivatives
(iv) Partial integration (integrating with respect to only one variable)
(v) Basic integration techniques and formulae
1 Partial differential equations

Definition 1.1. An equation involving an unknown function $z$ of two or more variables and its partial derivatives is called a partial differential equation (PDE).

1.1 Notations used

Throughout this module $z$ will be used to denote the unknown function $z(x, y)$ of the 2 independent variables $x, y$.

We will also use $p, q$ to denote the partial derivatives with respect to $x$, and $y$ respectively, i.e,

$$ p = \frac{\partial z}{\partial x}, \quad q = \frac{\partial z}{\partial y} $$

1.2 Some well known PDEs

The following are some well known partial differential equations from physics and engineering:

(i) Laplace’s equation

$$ \Delta z = z_{xx} + z_{yy} = 0 $$

(ii) Heat (or diffusion) equation

$$ z_t - \Delta z = 0 $$

(iii) Wave equation

$$ z_{tt} - \Delta z = 0 $$

(iv) Schrödinger’s equation

$$ iz_t + \Delta z = 0 $$

(v) Telegraph equation

$$ z_{tt} + dz_t - z_{xx} = 0 $$

(vi) Airy’s equation

$$ z_t + z_{xxx} = 0 $$

(vii) Beam equation

$$ z_{tt} + z_{xxxx} = 0 $$
1.3 Order and degree of a PDE

**Definition 1.2.** The order of the highest partial derivative in the equation is called the order of the PDE.

For example, \( z = x \frac{\partial z}{\partial x} + y \frac{\partial z}{\partial y} \) has order 1, and \( \frac{\partial^2 z}{\partial x \partial y} = \left( \frac{\partial z}{\partial x} \right)^3 \) has order 2.

**Definition 1.3.** The degree of the highest order partial derivative is called the degree of the PDE.

Both equations in the above example have degree 1. But \( z \left( \frac{\partial z}{\partial x} \right)^3 \) has degree 3.

## 2 Formation of partial differential equations

First we look at how to form a PDE from a given relation.

A relation involving 3 variables might include arbitrary constants or arbitrary functions. To form a PDE from such a relation, we have to eliminate the arbitrary quantities in the relation by taking the partial derivatives.

Partial differential equations can be formed using the following two methods from a relation \( f(x,y,z,a,b) = 0 \) involving three or more variables

(i) Eliminating arbitrary constants

(ii) Eliminating arbitrary function

When we get a relation involving arbitrary constants / functions, we don’t know what the order of the differential equation will be. This means we don’t know what order partial derivatives to take to eliminate the arbitrary constants / functions.

Luckily, we have a handy rule of thumb for how many times to differentiate:

(i) If the number of arbitrary constants equals the number of independent variables, then the PDE is of first order.

(ii) If the number of arbitrary constants is greater than the number of independent variables, then the PDE is of second or higher order.

(iii) If the PDE is formed by eliminating arbitrary functions, then the order of the PDE is in general equal to the number of arbitrary functions eliminated.
2.1 Elimination of arbitrary constants

For such problems, we find the partial derivatives of \( z \) to eliminate the arbitrary constants.

**Example 1.** Derive a PDE from the relation \( 2z = \frac{x^2}{a^2} + \frac{y^2}{b^2} \).

Solution: The given relation has 2 arbitrary constants \( a \) and \( b \). There are 2 independent variables \( x, y \). So by our rule of thumb, the resulting PDE should be of first order.

Find the first order partial derivatives of \( z \).

\[
2 \frac{\partial z}{\partial x} = \frac{2x}{a^2}, \quad 2 \frac{\partial z}{\partial y} = \frac{2y}{b^2}
\]

Simplifying the above expressions, we find

\[
p = \frac{\partial z}{\partial x} = \frac{x}{a^2}, \quad q = \frac{\partial z}{\partial y} = \frac{y}{b^2}
\]

Now we can write both arbitrary constants in terms of \( p, q, x, \) and \( y \) as

\[
\frac{1}{a^2} = \frac{p}{x}, \quad \frac{1}{b^2} = \frac{q}{y}
\]

Finally, substitute for the constants in the original relation to find

\[
2z = px + qy
\]

**Example 2.** Find the differential equation of all planes which are at a constant distance \( a \) from the origin.

Solution: A plane at distance \( a \) from the origin will have an equation of the form \( lx + my + nz = a \), where \( l, m, n \) are the direction cosines of the normal from the origin to the plane.

This relation has 3 arbitrary constants \( l, m, n \). But these constants are the direction cosines, and so satisfy the property : \( l^2 + m^2 + n^2 = 1 \), or \( n = \sqrt{1 - l^2 - m^2} \). So the equation of the plane can be written as

\[
lx + my + \sqrt{1 - l^2 - m^2} z = a
\]

Observe that the relation now has just 2 arbitrary constants, and 2 independent variables. So the PDE satisfied by this relation should be of first order.
Take the partial derivatives of (1) with respect to \(x\) and \(y\). We get the following equations:

\[
\begin{align*}
1 + \sqrt{1 - l^2 - m^2} &= 0 \quad (2) \\
1 + \sqrt{1 - l^2 - m^2} &= 0 \quad (3)
\end{align*}
\]

Next we eliminate the arbitrary constants \(l, m\) from equations (1), (2), (3). Squaring and adding (2) and (3),

\[
l^2 + m^2 = (1 - l^2 - m^2) (p^2 + q^2)
\]

Expand the right hand side and simplify to find

\[
l^2 + m^2 = (p^2 + q^2) - (l^2 + m^2) (p^2 + q^2)
\]

Hence,

\[
l^2 + m^2 = \frac{p^2 + q^2}{1 + p^2 + q^2}
\]

We can substitute this expression in (2), (3) to find \(l = -\frac{p}{\sqrt{1 + p^2 + q^2}}\), \(m = -\frac{q}{\sqrt{1 + p^2 + q^2}}\).

Finally, substitute for \(l, m, \sqrt{1 - l^2 - m^2}\) in (1)

\[
-\frac{px}{\sqrt{1 + p^2 + q^2}} - \frac{qy}{\sqrt{1 + p^2 + q^2}} + \frac{z}{\sqrt{1 + p^2 + q^2}} = a
\]

Rearranging this equation gives us the required PDE

\[
z = px + qy + a\sqrt{1 + p^2 + q^2}
\]

**Exercise**

Find PDEs from the following equations by eliminating arbitrary constants.

(i) \(z = ax + by + ab\)

(ii) \(z = ax + a^2 y^2 + b\)

(iii) \(z = (x^2 + a)(x^2 + b)\)

(iv) \(\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} + \frac{\partial^2 z}{\partial x \partial y} = 1\)

(v) \((x - a)^2 + (y - b)^2 = z^2 \cos^2 \alpha\), where \(\alpha\) is a fixed constant.
2.2 Elimination of arbitrary functions

While eliminating arbitrary functions also, we will directly take the partial derivatives of $z$. In problems where this is not possible, we will use the chain rule for partial derivatives.

**Example 3.** Form the PDE by eliminating arbitrary functions from $z = f(x + at) + g(x - at)$, where $a$ is a fixed constant.

Solution: In this relation, the independent variables are $x, t$ and there are 2 arbitrary functions $f, g$.

Again, by the rule of thumb, we have to differentiate as many times as the number of arbitrary functions - in this case 2.

Evaluate $\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial t}$.

$$\frac{\partial z}{\partial x} = f'(x + at) + g'(x - at), \quad \frac{\partial z}{\partial t} = af'(x + at) - ag'(x - at)$$

(By applying the chain rule)

$$\frac{\partial^2 z}{\partial x^2} = f''(x + at) + g''(x - at), \quad \frac{\partial^2 z}{\partial t^2} = a^2 f''(x + at) + a^2 g''(x - at)$$

Now by comparing the second order derivatives we can write

$$\frac{\partial^2 z}{\partial t^2} = a^2 \frac{\partial^2 z}{\partial x^2}$$

Note that this is just the wave equation from the examples.

**Example 4.** Form the PDE by eliminating arbitrary function from $z = (x+y)\phi(x^2 - y^2)$

Solution: The given relation has only 1 arbitrary function, so the required PDE should be of first order.

The partial derivatives of $z$ are:

$$p = \frac{\partial z}{\partial x} = (x + y)\phi'(x^2 - y^2) \cdot 2x + \phi(x^2 - y^2)$$

$$q = \frac{\partial z}{\partial y} = (x + y)\phi'(x^2 - y^2) \cdot (-2y) + \phi(x^2 - y^2)$$

(Using product rule and the chain rule)
From the original relation, we have \( \phi(x^2 - y^2) = \frac{z}{x+y} \). Substituting for \( \phi(x^2 - y^2) \) in the partial derivatives and rearranging gives us the following equations:

\[
p - \frac{z}{x+y} = 2(x+y)\phi'(x^2 - y^2)
\]

\[
q - \frac{z}{x+y} = -2y(x+y)\phi'(x^2 - y^2)
\]

Remember that to get the required PDE, we need to eliminate the arbitrary function \( \phi \). It is easy to see that dividing the above 2 equations will eliminate the \( \phi \) term.

The division gives

\[
\frac{p - \frac{z}{x+y}}{q - \frac{z}{x+y}} = \frac{x}{y}
\]

Taking the LCM and simplifying gives

\[
[x(x+y) - z]y + [q(x+y) - z]x = 0
\]

\[
(x+y)(py + qx) - z(x+y) = 0
\]

Dividing throughout by \( x+y \), the required PDE is

\[
py + qx = z
\]

**Example 5.** Form the PDE by eliminating arbitrary functions from the given relation: \( f(x^2 + y^2, z - xy) = 0 \)

**Solution:** For questions like these, we can’t evaluate the partial derivatives of \( z \) directly.

To solve, let \( u = x^2 + y^2, v = z - xy \), so that the relation is \( f(u, v) = 0 \).

Now apply the chain rule to find the partial derivative of the above function with respect to \( x \).

\[
\frac{\partial f}{\partial u} \left( \frac{\partial u}{\partial x} \frac{\partial x}{\partial z} + \frac{\partial u}{\partial z} \frac{\partial x}{\partial z} \right) + \frac{\partial f}{\partial v} \left( \frac{\partial v}{\partial x} \frac{\partial x}{\partial z} + \frac{\partial v}{\partial z} \frac{\partial x}{\partial z} \right) = 0
\]

If we now substitute for the appropriate partial derivatives of \( u, v \) and let \( p = \frac{\partial f}{\partial z} \), then

\[
\frac{\partial f}{\partial u}(2x) + \frac{\partial f}{\partial v}(-y + p) = 0
\]

\[\text{(4)}\]
Similarly, we can evaluate the partial derivative with respect to $y$ and get the equation

$$\frac{\partial f}{\partial y}(2y) + \frac{\partial f}{\partial x}(-x + q) = 0 \quad (5)$$

Now that we have 2 equations involving $\frac{\partial f}{\partial u}, \frac{\partial f}{\partial v}$, we can eliminate these partial derivatives from the equations:

Multiply the first equation by $y$ and the second equation by $x$. If we then subtract these 2 new equations, we find

$$\frac{\partial f}{\partial v}(-y^2 + py + x^2 - qx) = 0$$

Then either $\frac{\partial f}{\partial v} = 0$, or $(-y^2 + py + x^2 - qx) = 0$. But the partial derivative cannot identically be equal to zero (as $f$ is arbitrary, we cannot say anything specific about its partial derivatives).

Therefore,

$$(-y^2 + py + x^2 - qx) = 0$$

So the required PDE is

$$qx - py = x^2 - y^2$$

**Exercise**

Form PDEs from the following equations by eliminating arbitrary functions.

(i) $z = f_1(x) f_2(y)$

(ii) $z = e^{xy} \phi(x - y)$

(iii) $z = y^3 = 2f \left( \frac{1}{x} + \log y \right)$

(iv) $z = x f_1(x + t) + f_2(x + t)$

(v) $z = f \left( \frac{1}{x} \right)$

3 Solution of Partial Differential Equations

**Definition 3.1.** The solution $f(x,y,z,a,b) = 0$ of a first order differential equation which contains two arbitrary constants is called a complete integral.
Definition 3.2. A solution obtained from the complete integral which contains by assigning particular values to the arbitrary constants is called a particular Integral.

Methods for Solution of Partial Differential Equations

- **Method 1**: Equations Solvable by Direct Integration
- **Method 2**: Linear Equations of the First Order (Lagrange’s Linear Equation)
- **Method 3**: Non Linear Equations of the First Order (Charpit’s Method)
- **Method 4**: Method of Separation of Variables

3.1 Method 1

Equations Solvable by Direct Integration

In this method PDE can be solved by direct integration and in place of the usual constant of integration we use arbitrary functions of the variable held fixed.

Example 6. Solve \( \frac{\partial^2 z}{\partial x^2 \partial y} + 18xy^2 + \sin(2x - y) = 0 \)

Solution: Integrating the given PDE with respect to \( x \) (keeping \( y \) fixed)

\[ \frac{\partial^2 z}{\partial x \partial y} + 9x^2y^2 - \frac{1}{2} \cos(2x - y) = f(y) \]  
\[ \text{(6)} \]

[For the constant of integration, we use arbitrary functions of variable held fixed]

Integrating the equation (1) with respect to \( x \) (keeping \( y \) fixed)

\[ \frac{\partial z}{\partial y} + 3x^3y^2 - \frac{1}{4} \sin(2x - y) = xf(y) + g(y) \]  
\[ \text{(7)} \]

Integrate equation (2) with respect to \( y \) (keeping \( x \) fixed)

\[ z + x^3y^3 - \frac{1}{4} \cos(2x - y) = x \int f(y)dy + \int g(y)dy + w(x) \]  
\[ \text{(8)} \]
Then we substitute $\int f(y) \, dy = u(y)$ and $\int g(y) \, dy = v(y)$

$$z + x^3 y^3 - \frac{1}{4} \cos(2x - y) = xu(y) + v(y) + w(x) \quad (9)$$

$$\Rightarrow \quad z = \frac{1}{4} \cos(2x - y) - x^3 y^3 + xu(y) + v(y) + w(x)$$

where $u, v$ and $w$ are arbitrary functions.

**Example 7.** Solve $\frac{\partial^2 z}{\partial x^2} = xy$

Solution: Integrate the given PDE with respect to $x$ (keeping $y$ fixed)

$$\frac{\partial z}{\partial x} = \frac{x^2}{2} y + f(y) \quad (10)$$

Integrate equation (5) with respect to $x$ (keeping $y$ fixed)

$$z = \frac{x^3}{6} y + f(y) \int dx + g(y) \quad (11)$$

$$\Rightarrow \quad z = \frac{1}{6} x^3 y + xf(y) + g(y)$$

where $f$ and $g$ are arbitrary functions.

**Example 8.** Solve $\frac{\partial^2 z}{\partial x^2} + z = 0$, given that when $x = 0$, $z = e^y$ and $\frac{\partial z}{\partial x} = 1$

Solution: Here $z$ is a function of $x$ only. So we can treat the given PDE as ordinary differential equation (ODE) but constants can depend on $y$.

$$\Rightarrow \quad (D^2 + 1)z = 0$$

Auxiliary equation is $m^2 + 1 = 0 \Rightarrow m = i, -i$

$$\Rightarrow \quad z = f(y) \sin x + g(y) \cos y \quad (12)$$

Put $x = 0$ in the equation (7) (Applying the given conditions)

$$\Rightarrow \quad e^y = f(y) \sin 0 + g(y) \cos 0$$
\( e^y = g(y) \) and substitute in equation (7)

\[ \Rightarrow z = f(y) \sin x + e^y \sin y \]  \hspace{1cm} (13)

Put \( \frac{\partial z}{\partial x} = 1 \) when \( x = 0 \) in equation (8)

\[ \frac{\partial z}{\partial x} = f(y) \cos x + 0 \]  \hspace{1cm} (14)

When \( x = 0 \Rightarrow 1 = f(y) \)

Hence the desired solution is

\[ z = f(y) \sin x + g(y) \cos y \]  \hspace{1cm} (15)
\[ z = \sin x + e^y \cos y \]  \hspace{1cm} (16)

**Example 9.** Solve \( \frac{\partial^2 z}{\partial x^2 \partial y} = \cos (2x + 3y) \)

Solution: Integrating the given PDE with respect to \( y \) (keeping \( x \) fixed)

\[ \frac{\partial^2 z}{\partial x^2} = \frac{1}{3} \sin (2x + 3y) + f(x) \]  \hspace{1cm} (17)

Integrating the equation (12) with respect to \( x \)

\[ \frac{\partial z}{\partial x} = -\frac{1}{6} \cos (2x + 3y) + \int f(x)dx + g(y) \]  \hspace{1cm} (18)

Integrating the equation (13) with respect to \( x \)

\[ z = -\frac{1}{12} \sin (2x + 3y) + \int \int f(x)dxdx + g(y) \int dx + h(y) \]  \hspace{1cm} (19)

\[ z = -\frac{1}{12} \sin (2x + 3y) + u(x) + xg(y) + h(y) \]  \hspace{1cm} (20)

where \( u(x) = \int \int f(x)dxdx \) is an arbitrary function of \( x \)

**Exercise**

Solve the following equations by direct integration:
\( \frac{\partial^2 z}{\partial x \partial y} = \frac{x}{y} + a \)

\( \frac{\partial^2 z}{\partial x^2} = a^2 z \) given that when \( x = 0 \), \( \frac{\partial z}{\partial x} = \sin y \) and \( \frac{\partial z}{\partial y} = 0 \)

\( \frac{\partial^2 z}{\partial x \partial t} = e^{-t \cos x} \)

\( \frac{\partial^2 z}{\partial y^2} = z \), given that when \( y = 0 \), \( z = e^x \) and \( \frac{\partial z}{\partial y} = e^{-x} \)

\( \frac{\partial^2 z}{\partial x \partial y} = x^2 y \)

### 3.2 Method 2

**Linear Equations of the First Order**

**Definition 3.3.** If the unknown function and its partial derivatives appear to the power of 1 in a PDE then it is called *First Order Linear Partial Differential Equation*. Otherwise it is called non-linear PDE.

**Remark 1.** If \( z = f(x, y) \) then we shall employ the following notation for the partial derivatives.

\[ \frac{\partial z}{\partial x} = p, \quad \frac{\partial z}{\partial y} = q, \quad \frac{\partial^2 z}{\partial x^2} = r, \quad \frac{\partial^2 z}{\partial x \partial y} = s, \quad \frac{\partial^2 z}{\partial y^2} = t \]

**Example 10.**

(i) \( px + yq = 3z \)- First order linear PDE

(ii) \( xp - yq = y^2 - x^2 \)- First order linear PDE

(iii) \( p^2 + q^2 = 1 \)- Non linear
(iv) \( pq + p + q = 0 \)-Non linear

**Definition 3.4.** A linear PDE of the form \( Pp + Qq = R \) commonly
known as Lagrange’s linear equation where \( P, Q \) and \( R \) are functions
of \( x, y \) and \( z \)

**Procedure for the solution of** \( Pp + Qq = R \)

To solve the equation \( Pp + Qq = R \)

**Step 1** Identify the terms \( P, Q \) and \( R \) from \( Pp + Qq = R \)

**Step 2** Form the subsidiary equations (Auxiliary equation)

\[
\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}
\]

**Step 3** Solve the auxiliary equation by the following method to get two
independent solutions \( u = a \) and \( v = b \) where \( a \) and \( b \) are arbitrary
constants

(i) Grouping or Rearranging

(ii) Method of Lagrange multipliers

(iii) Both (i) and (ii)

**Step 4** Write the complete solution as \( \phi(u, v) = 0 \) or \( u = f(v) \)

**Case 1. Grouping**

We may be able to group two fractions from auxiliary equation and cancel
the common terms to get an equation in only two variables. On integrat-
ing the differential equation in only two variables by any methods, we
shall obtain one of the relations $u$ in the general solution. This method may be repeated to give another relation $v$ with the help of another two fractions in auxiliary equation.

**Example 11.** Solve $\frac{y^2z}{x}p + xzq = y^2$

**Solution:**

$$\frac{y^2z}{x}p + xzq = y^2$$  \hspace{1cm} (21)

Equation (16) $\times x \Rightarrow y^2zp + x^2zq = xy^2$

**Step 1** $P = y^2z$, $Q = x^2z$, $R = xy^2$

**Step 2** Auxiliary equation:

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} \Rightarrow \frac{dx}{y^2z} = \frac{dy}{x^2z} = \frac{dz}{xy^2}$$

**Step 3** Solve the auxiliary equation by grouping

**Taking first two fractions**

$$\frac{dx}{y^2z} = \frac{dy}{x^2z} \Rightarrow x^2 dx = y^2 dy$$

On integrating $\frac{x^3}{3} = \frac{y^3}{3} + \frac{a}{3} \Rightarrow x^3 - y^3 = a$

Set $u = x^3 - y^3 = a$

**Taking first and third fractions**

$$\frac{dx}{y^2z} = \frac{dz}{xy^2} \Rightarrow xdx = zdz$$

On integrating $\frac{x^2}{2} = \frac{z^2}{2} + \frac{b}{2} \Rightarrow x^2 - z^2 = b$

Set $v = x^2 - z^2 = b$

**Step 4** General equation is $\phi(u, v) = 0$ or $u = f(v)$

i.e. $\phi(x^3 - y^3, x^2 - z^2) = 0$ or $x^3 - y^3 = f(x^2 - z^2)$

**Example 12.** Solve $p \tan x + q \tan y = \tan z$

**Solution:**

**Step 1** $P = \tan x$, $Q = \tan y$, $R = \tan z$
Step 2 Auxiliary equation: \[ \frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} \Rightarrow \frac{dx}{\tan x} = \frac{dy}{\tan y} = \frac{dz}{\tan z} \]

Step 3 Solve the auxiliary equation by grouping

**Taking first two fractions** \( \frac{dx}{\tan x} = \frac{dy}{\tan y} \)

\( \Rightarrow \cot x \, dx = \cot y \, dy \)

**Result**

\[ \int \cot x \, dx = \int \frac{\cos x}{\sin x} \, dx = \log (\sin x) + c \]

On integrating \( \Rightarrow \log \sin x = \log \sin y + a \)

\( \Rightarrow \log \sin x - \log \sin y = a \Rightarrow \log \frac{\sin x}{\sin y} = a \)

Set \( u = \log \frac{\sin x}{\sin y} = a \)

**Taking second and third fractions** \( \frac{dy}{\tan y} = \frac{dz}{\tan z} \)

On integrating \( \Rightarrow \log \sin y = \log \sin z + b \)

\( \Rightarrow \log \sin y - \log \sin z = b \Rightarrow \log \frac{\sin y}{\sin z} = b \)

Set \( v = \log \frac{\sin y}{\sin z} = b \)

Step 4 General equation is \( \phi(u, v) = 0 \) or \( u = f(v) \)

i.e. \( \phi \left( \log \frac{\sin x}{\sin y}, \log \frac{\sin y}{\sin z} \right) = 0 \) or \( \log \frac{\sin x}{\sin y} = f \left( \log \frac{\sin y}{\sin z} \right) \)

**Example 13.** Solve \((p - q)z = z^2 + (x + y)^2\)

Solution:- Given PDE can be rearranged as \(pz - qz = z^2 + (x + y)^2\)

Step 1 \( P = z, \ Q = z, \ R = z^2 + (x + y)^2\)

Step 2 Auxiliary equation: \[ \frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} \Rightarrow \frac{dx}{z} = \frac{dy}{-z} = \frac{dz}{z^2 + (x + y)^2} \]
**Step 3** Solve the auxiliary equation by grouping

Taking first two fractions $\frac{dx}{z} = \frac{dy}{-z}$

$\Rightarrow dx = -dy$

On integrating $\Rightarrow x = -y + a = x + y = a$

set $u = x + y = a$

Taking first and third fractions $\frac{dx}{z} = \frac{dz}{z^2 + (x + y)^2}$

put $x + y = a$ in the above equation

$\Rightarrow \frac{dx}{z} = \frac{dz}{z^2 + a^2}$

$\Rightarrow \frac{dx}{z} = \frac{dz}{z^2 + a^2} \Rightarrow \frac{dx}{2z^2 + a^2} = \frac{1}{2z} dz$

**Result**

$$\int \frac{f'(x)}{f(x)} dx = \log f(x) + c$$

On integrating $\Rightarrow x = \frac{1}{2} \log (z^2 + a^2) + b'$

$\Rightarrow 2x = \log (z^2 + a^2) + b$

$\Rightarrow 2x - \log (z^2 + (x + y)^2) = b \quad [\text{since } x + y = a]$

set $u = 2x - \log (z^2 + (x + y)^2) = b$

**Step 4** General equation is $\phi(u, v) = 0$ or $u = f(v)$

i.e. $\phi(x + y, 2x - \log (z^2 + (x + y)^2)) = 0$ or

$x + y = f(2x - \log (z^2 + (x + y)^2))$
Note

Properties of ratio

(i) If \( \frac{a}{b} = \frac{c}{d} = \frac{e}{f} \) then \( \frac{a+c+e}{b+d+f} = \frac{a}{b} = \frac{c}{d} = \frac{e}{f} \)

(ii) \( \frac{a}{b} = \frac{c}{d} = \frac{e}{f} \) then \( \frac{la+mc+ne}{lb+md+nf} = \frac{a}{b} = \frac{c}{d} = \frac{e}{f} \) where \( l, m \) and \( n \) are constants or variables.

Example 14. Solve \((x^2 - y^2 - z^2)p + 2xyq = 2xz\)

Solution:-

Step 1 \( P = x^2 - y^2 - z^2, \ Q = 2xy, \ R = 2xz \)

Step 2 \( \frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} \Rightarrow \frac{dx}{x^2 - y^2 - z^2} = \frac{dy}{2xy} = \frac{dz}{2xz} \)

Step 3 Solve the equation in step 2 by grouping

Taking second and third fractions \( \frac{dy}{2xy} = \frac{dz}{2xz} \)

\[ \Rightarrow \frac{dy}{y} = \frac{dz}{z}, \text{ on integrating } \Rightarrow \log y = \log z + \log a \]

\[ \Rightarrow \log y - \log z = \log a \Rightarrow \frac{y}{z} = \log a \]

\[ \Rightarrow \frac{y}{z} = a. \text{ Set } u = \frac{y}{z} = a \]

Apply the property of ratio

Multiply each ratio in auxiliary equation by \( x, y \) and \( z \) respectively

\[ i.e., \frac{xdx + ydy + zdz}{x^3 - xy^2 - xz^2 + 2xy^2 + 2xz^2} = \frac{xdx + ydy + zdz}{x^3 + xy^2 + xz^2} = \frac{xdx + ydy + zdz}{x(x^2 + y^2 + z^2)} \]

Equating the new ratio in to third one

\[ \frac{xdx + ydy + zdz}{x(x^2 + y^2 + z^2)} = \frac{dz}{2xz} \Rightarrow \frac{2(xdx + ydy + zdz)}{x^2 + y^2 + z^2} = \frac{dz}{z} \]
\[ \int \frac{f(x)}{f(x)} \, dx = \log f(x) + c \]

On integrating \( \Rightarrow \log(x^2 + y^2 + z^2) = \log z + \log b \)
\[ \Rightarrow \log(x^2 + y^2 + z^2) - \log z = \log b \]
\[ \Rightarrow \log \left( \frac{x^2 + y^2 + z^2}{z} \right) = \log b \]
\[ \Rightarrow \frac{x^2 + y^2 + z^2}{z} = b \Rightarrow \text{set } v = \frac{x^2 + y^2 + z^2}{z} = b \]

\textbf{Step 4} General equation is \( \phi(u, v) = 0 \) or \( u = f(v) \)
i.e. \( \phi \left( \frac{y}{z}, \frac{x^2 + y^2 + z^2}{z} \right) = 0 \) or \( \frac{y}{z} = f \left( \frac{x^2 + y^2 + z^2}{z} \right) \)

\textbf{Example 15.} Solve \((x^2 - yz)p + (y^2 - zx)q = z^2 - xy\)

\textbf{Solution:-}

\textbf{Step 1} \( P = x^2 - yz, \ Q = y^2 - zx, \ R = z^2 - xy \)

\textbf{Step 2} Auxiliary equation- \( \frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} \)
\[ \Rightarrow \frac{dx}{x^2 - yz} = \frac{dy}{y^2 - zx} = \frac{dz}{z^2 - xy} \]

\textbf{Step 3} Solve the auxiliary equation using the properties of ratios by grouping

\textit{Taking the difference between first two and last two fractions and equating them}

\[ \frac{dx - dy}{x^2 - yz - (y^2 - zx)} = \frac{dy - dz}{y^2 - z^2 - zx + xy} \]
\[ \Rightarrow \frac{dx - dy}{x^2 - y^2 - yz + zx} = \frac{dy - dz}{y^2 - z^2 - zx + xy} \]
\[ \Rightarrow \frac{dx - dy}{(x + y)(x - y) + z(x - y)} = \frac{dy - dz}{(y + z)(y - z) + x(y - z)} \]
\[
\Rightarrow \frac{dz - dy}{(x+y)(x-y) + z(x-y)} = \frac{dy - dz}{(y+z)(y-z) + x(y-z)}
\]
\[
\Rightarrow \frac{dx - dy}{(x-y)(x+y+z)} = \frac{dy - dz}{(y-z)(x+y+z)}
\]
\[
\Rightarrow \frac{dx - dy}{(x-y)} = \frac{dy - dz}{(y-z)}
\]
On integrating, \( \log (x-y) = \log (y-z) + \log a \)
\[
\Rightarrow \log (x-y) - \log (y-z) = \log a
\]
\[
\Rightarrow \log \frac{x-y}{y-z} = \log a \Rightarrow \frac{x-y}{y-z} = a
\]
Set \( u = \frac{x-y}{y-z} = a \)

**Taking the difference between last two and last and first fractions and equating them**
\[
\frac{dy - dz}{y^2 - zx - (z^2 - xy)} = \frac{dx - dz}{z^2 - xy - (x^2 - yz)}
\]
\[
\Rightarrow \frac{dy - dz}{y^2 - z^2 - zx + xy} = \frac{dx - dz}{z^2 - x^2 - xy + yz}
\]
\[
\Rightarrow \frac{dy - dz}{(y+z)(y-z) + x(y-z)} = \frac{dx - dz}{(z+x)(z-x) + y(z-x)}
\]
\[
\Rightarrow \frac{dy - dz}{(y-z)(x+y+z)} = \frac{dx - dz}{(z-x)(x+y+z)}
\]
On integrating, \( \log (y-z) = \log(z-x) + \log b \)
\[
\Rightarrow \log (y-z) - \log(z-x) = \log b
\]
\[
\Rightarrow \log \frac{y-z}{z-x} = \log b \Rightarrow \frac{y-z}{z-x} = b
\]
Set \( v = \frac{y-z}{z-x} = b \)

**Step 4** General equation is \( \phi(u,v) = 0 \) or \( u = f(v) \)

i.e. \( \phi \left( \frac{x-y}{y-z}, \frac{y-z}{z-x} \right) = 0 \) or \( \frac{x-y}{y-z} = f \left( \frac{y-z}{z-x} \right) \)
Example 16. Solve \( y^2 p - xy q = x(z - 2y) \)

Solution:- Given PDE can be rearranged as \( y^2 p - xyq = xz - 2xy \)

Step 1 \( P = y^2, \ Q = -xy, \ R = xz - 2xy \)

Step 2 Auxiliary equation- \( \frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} \Rightarrow \frac{dx}{y^2} = \frac{dy}{-xy} = \frac{dz}{xz - 2xy} \)

Step 3 Solve the auxiliary equation by grouping

Taking first two fractions \( \frac{dx}{y^2} = \frac{dy}{-xy} \)

\[ \Rightarrow \frac{dx}{y} = \frac{dy}{-x} \Rightarrow xdx = -ydy \]

On integrating \( \Rightarrow \frac{x^2}{2} = -\frac{y^2}{2} + \frac{a}{2} \Rightarrow x^2 + y^2 = a \)

Set \( u = x^2 + y^2 = a \)

Taking second and third fractions \( \frac{dy}{-xy} = \frac{dz}{x(z - 2y)} \)

\[ \Rightarrow \frac{dy}{y} = \frac{dz}{(z - 2y)} \Rightarrow (z - 2y)dy = -ydz \]

\[ \Rightarrow zdy - 2ydy = -ydz \Rightarrow zdy + ydz = 2ydy \]

Result: Product rule for differentiation- \( d(uy) = u \, dv + v \, du \)

\[ \Rightarrow d(zy) = 2ydy \]

On integrating \( zy = 2\frac{y^2}{2} + a \Rightarrow zy - y^2 = a \)

Step 4 General equation is \( \phi(u, v) = 0 \) or \( u = f(v) \)

i.e. \( \phi(x^2 + y^2, zy - y^2) = 0 \) or \( x^2 + y^2 = f(zy - y^2) \)

Exercise

Solve the following PDE

(i) \( xp + yq = 3z \)
\( (ii) \ p \sqrt{x} + q \sqrt{y} = z \)

\( (iii) \ y^2 zp + x^2 zq = y^2 x \)

\( (iv) \ p yz + q zx = x y \)

\( (v) \ xy - yq = y^2 - x^2 \)

\( (vi) \ p - q = \log (x + y) \)

\( (vii) \ p + 3q = 5z + \tan(y - 3x) \)

\( (viii) \ (y^2 + z^2 - x^2)p - 2xyq + 2xz = 0 \)

---

**Case 2.  Method of Lagrange’s Multipliers**

Let \( P_1, Q_1 \) and \( R_1 \) be the function of \( x, y \) and \( z \). Then by the property of ratios of each fraction in auxiliary equation \( \frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} \) will be equal to

\[
\frac{P_1 dx + Q_1 dy + R_1 dz}{PP_1 + QQ_1 + RR_1}
\]

If \( PP_1 + QQ_1 + RR_1 = 0 \), then \( P_1 dx + Q_1 dy + R_1 dz = 0 \). This equation \( P_1 dx + Q_1 dy + R_1 dz = 0 \) can be integrated to give one of the solution \( u = a \). This method may be repeated to get another solution \( v = b \).

**Remark 2.** \( P_1, Q_1 \) and \( R_1 \) are called Lagrange’s multipliers. As a special case, Lagrange’s multipliers can be constants also.

**Working Rule for Lagrange’s Multipliers**
(i) Choose \( P_1, Q_1 \) and \( R_1 \) as Lagrange’s multipliers such that
\[
PP_1 + QQ_1 + RR_1 = 0
\]

(ii) Solve the differential equation \( P_1 dx + Q_1 dy + R_1 dz = 0 \) (by integration) for the solution \( u = a \)

(iii) The above procedure may be repeated for the second solution of auxiliary equation \( v = b \)

**Example 17.** Solve \( x(y - z)p + y(z - x)q = z(x - y) \)

**Solution:**

**Step 1** \( P = x(y - z), \quad Q = y(z - x), \quad R = z(x - y) \)

**Step 2** Auxiliary equation:
\[
\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}
\]
\[
\Rightarrow \quad \frac{dx}{x(y - z)} = \frac{dy}{y(z - x)} = \frac{dz}{z(x - y)}
\]

**Step 3** Solve the auxiliary equation by using Lagrange’s multipliers

Choose \( P_1 = \frac{1}{x}, \quad Q_1 = \frac{1}{y} \) and \( R_1 = \frac{1}{z} \) such that

\[
PP_1 + QQ_1 + RR_1 = x(y - z)\left(\frac{1}{x}\right) + y(z - x)\left(\frac{1}{y}\right) + z(x - y)\left(\frac{1}{z}\right)
\]
\[
= y - z + z - x + x - y
\]
\[
= 0
\]

So the solution \( u = a \) is the solution of the differential equation

\[
P_1 dx + Q_1 dy + R_1 dz = 0 \Rightarrow \frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz = 0
\]

On integrating, \( \log x + \log y + \log z = \log a \)

\[
\Rightarrow \log (xyz) = \log a \Rightarrow xyz = a
\]
Set \( u = xyz = a \)

Choose \( P_2 = 1, \ Q_2 = 1 \) and \( R_2 = 1 \) such that

\[
PP_2 + QQ_2 + RR_2 = x(y - z) + y(z - x) + z(x - y) = xy - xz + yz - yx + zx - zy = 0
\]

So the solution \( v = b \) is the solution of the differential equation

\[
P_2dx + Q_2dy + R_2dz = 0 \Rightarrow \frac{dx}{P} + \frac{dy}{Q} + \frac{dz}{R} = 0
\]

On integrating, \( x + y + z = b \)

\( \Rightarrow \) \text{ set } v = x + y + z = b

**Step 4** General equation is \( \phi(u, v) = 0 \) or \( u = f(v) \)

i.e. \( \phi(xyz, x + y + z) = 0 \) or \( xyz = f(x + y + z) \)

**Example 18.** Solve \( x(y^2 - z^2)p + y(z^2 - x^2)q - z(x^2 - y^2) = 0 \)

Solution:- Given PDE can be rearranged in standard form \( Pp + Qq = R \)

as \( x(y^2 - z^2)p + y(z^2 - x^2)q = z(x^2 - y^2) \)

**Step 1** \( P = x(y^2 - z^2), \ Q = y(z^2 - x^2), \ R = z(x^2 - y^2) \)

**Step 2** Auxiliary equation

\[
\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}
\]

\( \Rightarrow \frac{dx}{x(y^2 - z^2)} = \frac{dy}{y(z^2 - x^2)} = \frac{dz}{z(x^2 - y^2)} \)

**Step 3** Solve the auxiliary equation by using Lagrange’s multipliers

Choose \( P_1 = \frac{1}{x}, \ Q_1 = \frac{1}{y} \) and \( R_1 = \frac{1}{z} \) such that

\[
PP_1 + QQ_1 + RR_1 = x(y^2 - z^2)^{\frac{1}{x}} + y(z^2 - x^2)^{\frac{1}{y}} + z(x^2 - y^2)^{\frac{1}{z}}
\]

\( = y^2 - z^2 + z^2 - x^2 + x^2 - y^2 \)

\( = 0 \)
So the solution \( u = a \) is the solution of the differential equation

\[
P_1 dx + Q_1 dy + R_1 dz = 0 \Rightarrow \frac{1}{x} dx + \frac{1}{y} dy + \frac{1}{z} dz = 0
\]

On integrating, \( \log x + \log y + \log z = \log a \)

\[\Rightarrow \log (xyz) = \log a \Rightarrow xyz = a\]

Set \( u = xyz = a \)

Choose \( P_2 = x \), \( Q_2 = y \) and \( R_2 = z \) such that

\[
PP_2 + QQ_2 + RR_2 = x^2(y^2 - z^2) + y^2(z^2 - x^2) + z^2(x^2 - y^2)
\]

\[
= x^2y^2 - x^2z^2 + y^2z^2 - y^2x^2 + z^2x^2 - z^2y^2
\]

\[= 0\]

So the solution \( v = b \) is the solution of the differential equation

\[
P_2 dx + Q_2 dy + R_2 dz = 0 \Rightarrow x dx + y dy + z dz = 0
\]

On integrating, \( \frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = \frac{a}{2} \)

\[\Rightarrow x^2 + y^2 + z^2 = a \Rightarrow \text{set } u = x^2 + y^2 + z^2 = b\]

**Step 4** General equation is \( \phi(u,v) = 0 \) or \( u = f(v) \)

i.e. \( \phi(xyz, x^2 + y^2 + z^2) = 0 \) or \( xyz = f(x^2 + y^2 + z^2) \)

**Example 19.** Solve \( x^2(y - z)p + y^2(z - x)q = z^2(x - y) \)

Solution:

**Step 1** \( P = x^2(y - z) , \ Q = y^2(z - x) , \ R = z^2(x - y) \)

**Step 2** Auxiliary equation

\[
\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}
\]

\[\Rightarrow \frac{dx}{x^2(y - z)} = \frac{dy}{y^2(z - x)} = \frac{dz}{z^2(x - y)}\]
Step 3 Solve the auxiliary equation by using Lagrange’s multipliers

Choose \( P_1 = \frac{1}{x} \), \( Q_1 = \frac{1}{y} \) and \( R_1 = \frac{1}{z} \) such that

\[
P P_1 + Q Q_1 + R R_1 = x^2(y - z) \frac{1}{x} + y^2(z - x) \frac{1}{y} + z^2(x - y) \frac{1}{z}
\]

\[
= x(y - z) + y(z - x) + z(x - y)
\]

\[
= xy - xz + yz - yx + zx - zy
\]

\[
= 0
\]

So the solution \( u = a \) is the solution of the differential equation

\[
P_1dx + Q_1dy + R_1dz = 0 \Rightarrow \frac{1}{x}dx + \frac{1}{y}dy + \frac{1}{z}dz = 0
\]

On integrating, \( \log x + \log y + \log z = \log a \)

\( \Rightarrow \) \( \log (xyz) = \log a \) \( \Rightarrow \) \( xyz = a \)

Set \( u = xyz = a \)

Choose \( P_2 = \frac{1}{x^2} \), \( Q_2 = \frac{1}{y^2} \) and \( R_2 = \frac{1}{z^2} \) such that

\[
P P_2 + Q Q_2 + R R_2 = x^2(y - z) \frac{1}{x^2} + y^2(z - x) \frac{1}{y^2} + z^2(x - y) \frac{1}{z^2}
\]

\[
= y - z + z - x + x - y
\]

\[
= 0
\]

So the solution \( v = b \) is the solution of the differential equation

\[
P_2dx + Q_2dy + R_2dz = 0 \Rightarrow \frac{1}{x^2}dx + \frac{1}{y^2}dy + \frac{1}{z^2}dz = 0
\]

Result

\[
\frac{d}{dx} \left( \frac{1}{x} \right) = -\frac{1}{x^2} \quad \text{then} \quad \int \frac{1}{x^2}dx = -\frac{1}{x} + c
\]
On integrating, \(-\frac{1}{x} - \frac{1}{y} - \frac{1}{z} = -b\)
\[\Rightarrow \frac{1}{x} + \frac{1}{y} + \frac{1}{z} = b \Rightarrow \text{set } v = \frac{1}{x} + \frac{1}{y} + \frac{1}{z} = b\]

**Step 4** General equation is \(\phi(u,v) = 0\) or \(u = f(v)\)
\[\text{i.e. } \phi \left( xyz, \frac{1}{x} + \frac{1}{y} + \frac{1}{z} \right) = 0 \text{ or } xyz = f \left( \frac{1}{x} + \frac{1}{y} + \frac{1}{z} \right)\]

**Example 20.** Solve \(( mz - ny ) \frac{\partial z}{\partial x} + ( nx - lz ) \frac{\partial z}{\partial y} = ly - mx\)

**Solution:-**

**Step 1** \(P = ( mz - ny ), \ Q = ( nx - lz ), \ R = ( ly - mx )\)

**Step 2** Auxiliary equation-
\[\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}\]
\[\Rightarrow \frac{dx}{mx - ny} = \frac{dy}{nx - lz} = \frac{dz}{ly - mx}\]

**Step 3** Solve the auxiliary equation by using Lagrange’s multipliers

Choose \(P_1 = x, \ Q_1 = y \text{ and } R_1 = z\) such that
\[PP_1 + QQ_1 + RR_1 = x(mx - ny) + y(nx - lz) + z(ly - mx)\]
\[= mxz - nxy + nxy - lzy + lzy - mzx\]
\[= 0\]

So the solution \(u = a\) is the solution of the differential equation
\[P_1dx + Q_1dy + R_1dz = 0 \Rightarrow x \ dx + y \ dy + z \ dz = 0\]

On integrating, \[\frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = \frac{a}{2}\]
\[\Rightarrow x^2 + y^2 + z^2 = a \Rightarrow \text{set } u = x^2 + y^2 + z^2 = a\]
Choose $P_2 = l, \ Q_2 = m$ and $R_2 = n$ such that

\[ PP_2 + QQ_2 + RR_2 = (mz - ny)l + (nx - l)lm + (ly - mx)n \]
\[ = mlz - nly + mmx - bmx + bny - nmx \]
\[ = 0 \]

So the solution $v = b$ is the solution of the differential equation

\[ P_2dx + Q_2dy + R_2dz = 0 \Rightarrow \ ldx + ndy + ndz = 0 \]

On integrating, $lx + my + nz = b \Rightarrow \ set \ v = lx + my + nz = b$

**Step 4** General equation is $\phi(u, v) = 0$ or $u = f(v)$

i.e. $\phi(x^2 + y^2 + z^2, lx + my + nz) = 0$ or $x^2 + y^2 + z^2 = f(lx + my + nz)$

**Exercise**

Solve the following PDE

(i) $x(z^2 - x^2)p + y(x^2 - z^2)q = z(y^2 - z^2)$

(ii) $z(x + y)p + z(x - y)q = x^2 + y^2$ (Hint: $(x, -y, z)$ and $(y, z, -x)$ are the multipliers)

(iii) $(z - y)p + (x - z)q = y - x$

**Case 3. Combination of Case 1 and Case 2**

We can solve auxiliary equation of a linear PDE by using case 1 and case 2 for $u = a$ and $v = b$. 
Example 21. Solve \((z^2 - 2yz - y^2)p + (xy + zx)q = xy - zx\)

Solution:-

Step 1 \(P = z^2 - 2yz - y^2, \ Q = xy + yz, \ R = xy - zx\)

Step 2 Auxiliary equation- \(\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}\)

\[\Rightarrow \frac{dx}{z^2 - 2yz - y^2} = \frac{dy}{xy + zx} = \frac{dz}{xy - zx}\]

Step 3 Solve the auxiliary equation

Calculation of \(u = a\) by using Lagrange’s multipliers

Choose \(P_1 = x, \ Q_1 = y\) and \(R_1 = z\) such that

\[PP_1 + QQ_1 + RR_1 = x(z^2 - 2yz - y^2) + y(xy + zx) + z(xy - zx)\]
\[= z^2x - 2xyz - xy^2 + xy^2 + zxy + zxy - z^2x\]
\[= 0\]

So the solution \(u = a\) is the solution of the differential equation

\[P_1dx + Q_1dy + R_1dz = 0 \Rightarrow x \ dx + y \ dy + z \ dz = 0\]

On integrating,

\[\frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = a\]

\[\Rightarrow x^2 + y^2 + z^2 = a \Rightarrow \text{set} \ u = x^2 + y^2 + z^2 = a\]

Calculation of \(v = b\) by grouping [Taking last two fractions]

\[\frac{dy}{xy + zx} = \frac{dz}{xy - zx}\]

\[\Rightarrow \frac{dy}{x(y + z)} = \frac{dz}{x(y - z)} \Rightarrow \frac{dy}{y + z} = \frac{dz}{y - z}\]

\[\Rightarrow (y - z)dy = (y + z)dz \Rightarrow ydy - zdy = ydz + zdz\]

\[\Rightarrow ydy - zdz = ydz + zdy\]
$dy - zdz = d(yz)$

On integrating, $\frac{y^2}{2} - \frac{z^2}{2} = yz + b \Rightarrow \frac{y^2}{2} - \frac{z^2}{2} - yz = b$

Set $v = \frac{y^2}{2} - \frac{z^2}{2} - yz = b$

**Step 4** General equation is $\phi(u,v) = 0$ or $u = f(v)$

i.e. $\phi \left( x^2 + y^2 + z^2, \frac{y^2}{2} - \frac{z^2}{2} - yz \right) = 0$ or $x^2 + y^2 + z^2 = f \left( \frac{y^2}{2} - \frac{z^2}{2} - yz \right)$

**Exercise**

Solve the following PDE.

(i) $\left( \frac{y - z}{yz} \right) p + \left( \frac{z - x}{zx} \right) q = \frac{x - y}{xy}$

(ii) $z(xp - yq) = y^2 - x^2$

(iii) $x^2 p + y^2 q = (x + y)z$

3.3 **Method 3**

**Non Linear Equations of the First Order (CHARPIT’S METHOD)**

**Definition 3.5.** A PDE which involves first order partial derivatives $p$ and $q$ with degree higher than one and the product of $p$ and $q$ is called a non linear partial differential equation of the first order.

**Example:** $p^2 + q^2 = z, \; y p + x q + pq = 0$
CHARPIT’S METHOD

This is a general method for solving non-linear first order partial differential equation \( f(x, y, z, p, q) = 0 \) with two independent variables. The complete solution of such an equation contains only two arbitrary constant. (equal to the number of independent variables \( x \) and \( y \))

Working Rule for the Solution of \( f(x, y, z, p, q) = 0 \) in Charpit’s Method

Step 1 Rearrange the PDE in the form \( f(x, y, z, p, q) = 0 \)

Step 2 Find out the following Partial derivatives
\[
\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}, \frac{\partial f}{\partial p} \text{ and } \frac{\partial f}{\partial q}
\]

Step 3 Write down the Charpit’s auxiliary equation
\[
\frac{dx}{fp} = \frac{dy}{fq} = \frac{dz}{(pf + qf)z} = \frac{dp}{-(fz + pf)} = \frac{dq}{-(fy + qfz)}
\]

Step 4 Solve the above auxiliary equation by grouping or rearrangement of any two fractions and derive simplest relation involving at least one of \( p \) and \( q \).

Step 5 The simplest relation along with the given PDE to determine \( p \) and \( q \)

Step 6 Substitute the value of \( p \) and \( q \) in \( dz = pdx + qdy \) [since \( z = f(x, y) \Rightarrow \) complete derivative is \( dz = pdx + qdy \)]

Step 7 On integration of the above equation gives the complete integral of the given equation
Example 22. Solve $(p^2 + q^2)y = qz$

Solution:-

Step 1 Rearrange the given PDE as $f(x, y, z, p, q) = 0$
\[ \Rightarrow f = p^2y + q^2y - qz = 0 \]

Step 2 Find the first order partial derivatives
\[ f_x = 0, \ f_y = p^2 + q^2, \ f_z = -q, \ f_p = 2pq \text{ and } f_q = 2qy - z \]

Step 3 Write down the Charpit’s auxiliary equation
\[
\frac{dx}{f_p} = \frac{dy}{f_q} = \frac{dz}{(pf_x + qf_y)} = \frac{dp}{-(f_x + pf_z)} = \frac{dq}{-(f_y + qf_z)} \tag{22}
\]
\[
pf_y + qf_x = p \times 2pq + q \times (2qy - z) = 2p^2y + 2q^2y - qz
\]
\[ = 2(p^2 + q^2)y - qz \]
\[ = 2qz - qz \ [\text{From given PDE } (p^2 + q^2)y = qz] \]
\[ = qz \]
\[
f_x + pf_z = 0 + p(-q) = -pq
\]
\[
f_y + qf_x = p^2 + q^2 + q(-q) = p^2 + q^2 - q^2 = p^2
\]
Substitute these values in equation (17)
\[
\frac{dx}{2pq} = \frac{dy}{2qy - z} = \frac{dz}{(qz)} = \frac{dp}{-(pq)} = \frac{dq}{-(p^2)}
\]
\[
\frac{dx}{2pqy} = \frac{dy}{2qy - z} = \frac{dz}{qz} = \frac{dp}{-pq} = \frac{dq}{p^2}
\]

Step 4 Taking last two fractions of above auxiliary
\[
\frac{dp}{-pq} = \frac{dq}{p^2}
\]
\[ \Rightarrow pdp = -qdq \Rightarrow pdp + qdq = 0 \]
On integrating, \[ \frac{p^2}{2} + \frac{q^2}{2} = \frac{c^2}{2} \Rightarrow p^2 + q^2 = c^2 \]
Step 5 Substitute the value of the equation \( p^2 + q^2 = c^2 \), from above step, in the given PDE \((p^2 + q^2)y = qz\)

\[
\Rightarrow c^2y = qz \Rightarrow q = \frac{c^2y}{z}
\]

Substitute the value of \( q \) in the equation \( p^2 + q^2 = c^2 \)

\[
\Rightarrow, \quad p^2 + \left(\frac{c^2y^2}{z^2}\right) = c^2 \Rightarrow \quad p^2\frac{z^2}{z^2} + \frac{c^4y^2}{z^2} = c^2 \Rightarrow \quad p^2z^2 + c^4y^2 = c^2z^2
\]

\[
p^2z^2 = c^2z^2 - c^4y^2 \Rightarrow p = \sqrt{\left[\frac{c^2z^2 - c^4y^2}{z^2}\right]} \Rightarrow p = \frac{c\sqrt{z^2 - c^4y^2}}{z}
\]

Step 6 Substitute the value of \( p \) and \( q \) in the equation \( dz = pdx + qdy \)

\[
\Rightarrow dz = \frac{c\sqrt{z^2 - c^4y^2}}{z} dx + \frac{c^2y}{z} dy
\]

\[
\Rightarrow dz = \frac{c\sqrt{z^2 - c^4y^2}dx}{z^2} + \frac{c^2ydy}{z^2}
\]

\[
zdz - c^2ydy = c\sqrt{z^2 - c^4y^2}dx
\]

\[
\Rightarrow \frac{zdz - c^2ydy}{\sqrt{z^2 - c^4y^2}} = c \ dx
\]

\[
\text{[put } t = z^2 - c^2y^2 \Rightarrow dt = 2zdz - c^22y \ dy \Rightarrow dt = 2(zdz - c^2ydy)\]
\]

\[
\Rightarrow \frac{dt}{2\sqrt{t}} = c \ dx \quad \left(\frac{d}{dx}(\sqrt{z}) = \frac{1}{2\sqrt{x}} \Rightarrow \int \frac{1}{2\sqrt{x}} = \sqrt{x} + c\right)
\]

Step 7 On integrating, \( \sqrt{t} = cx + a \) \( \Rightarrow \sqrt{z^2 - c^2y^2} = cx + a \)

\[
z^2 - c^2y^2 = (cx + a)^2
\]

\[
z^2 = c^2y^2 + (cx + a)^2 \text{ which is the required complete integral.}
\]

Example 23. Solve \( 2zx - pxz^2 - 2qxy + pq = 0 \)

Solution:

Step 1 \( f(x, y, z, p, q) = 2zx - pxz^2 - 2qxy + pq = 0 \)
step 2 $f_x = 2z - 2xp - 2qy, \ f_y = -2qx, \ f_z = 2x, \ f_p = -x^2 + q, \ f_q = -2xy + p$

**Step 3** Write down the Charpits’s auxiliary equation

$$\frac{dx}{f_p} = \frac{dy}{f_q} = \frac{dz}{(pf_p + qf_q)} = \frac{dp}{-(f_x + pf_z)} = \frac{dq}{-(f_y + qf_z)}$$

$$pf_p + qf_q = p(-x^2 + q) + q(-2xy + p)$$
$$= -px^2 + pq - 2xyq + pq$$
$$= 2pq - px^2 - 2xyq$$

$$f_x + pf_z = 2z - 2xp - 2qy + p, 2x = 2z - 2xp - 2qy + 2xp = 2z - 2qy$$

$$f_y + qf_z = -2qx + q, 2x = -2qx + 2qx = 0$$

Substitute these values in the above equation

$$\frac{dx}{-x^2 + q} = \frac{dy}{-2xy + p} = \frac{dz}{2pq - px^2 - 2xyq} = \frac{dp}{-(2z - 2qy)} = \frac{dq}{0}$$

**Step 4** From the last fraction, $\frac{dq}{0} \Rightarrow dq = 0 \Rightarrow q = a$

**Step 5** Put $q = a$ in the given PDE $\Rightarrow 2zx - px^2 - 2axy + pa = 0$

$\Rightarrow pa - px^2 = 2axy - 2zx$

$\Rightarrow p(a - x^2) = 2x(ay - z)$

$\Rightarrow p = \frac{2x(ay - z)}{a - x^2}$

**Step 6** Substitute the value of $p$ and $q$ in $dz = pdx + qdy$

$\Rightarrow dz = \frac{2x(ay - z)}{a - x^2} dx + ady$

$\Rightarrow dz - ady = \frac{2x(ay - z)}{a - x^2} dx$

$\Rightarrow \frac{dz - ady}{ay - z} = \frac{2x}{a - x^2} dx$

$\Rightarrow \frac{dz - ady}{z - ay} = \frac{2x}{x^2 - a} dx$

{Take negative sign common from denominator}
Step 7 on integrating \( \int \frac{dz - ady}{ay - z} = \int \frac{2x}{a - x^2} dx \)

Result
\[
\int \frac{f(x)}{f(x)} dx = \log f(x) + x
\]

\( \Rightarrow \) \( \log (z - ay) = \log (x^2 - a) + \log b \)

\( \Rightarrow \) \( \log (z - ay) - \log (x^2 - a) = \log b \)

\( \Rightarrow \) \( \log \frac{z - ay}{x^2 - a} = \log b \Rightarrow \frac{z - ay}{x^2 - a} = b \)

\( \Rightarrow \) \( z - ay = b(x^2 - a) \)

\( \Rightarrow \) \( z = ay + b(x^2 - a) \)

3.4 Method 4

Method of Separation of Variables

The differential equation together with some specified conditions (boundary conditions) constitute boundary value problem. Most of the boundary value problems involving linear PDE can be solved by the method of separation of variables. This method involves a solution which break up into a product of functions each of which contains only one of the variables

Example 24. Solve by the method of separation of variables
\[
\frac{\partial^2 z}{\partial x^2} - 2 \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = 0
\]

Solution:- Assume the solution of given Partial differential equation is
\[
z = XY
\]
where $X$ is a function of $x$ alone and $Y$ that of $y$ alone. [From the given PDE, the dependent variable $z$ is partially differentiate w.r.to $x$ and $y$. So in this method, solution is a product of functions of $x$ and $y$.]

If $z = XY$, then 
\[
\frac{\partial z}{\partial x} = X'Y, \quad \frac{\partial z}{\partial y} = XY', \quad \frac{\partial^2 z}{\partial x^2} = X''Y
\]

Substitute these values in given PDE
\[
X''Y - 2X'Y + XY' = 0
\]
\[
X''Y - 2X'Y = XY'
\]
\[
(X'' - 2X')Y = -XY'
\]
\[
\frac{X'' - 2X'}{X} = \frac{Y'}{Y}
\]

Since $x$ and $y$ are independent variables, it can only be true if each side is equal to the same constant, $a$ (say)
\[
\frac{X'' - 2X'}{X} = a \quad \text{and} \quad \frac{Y'}{Y} = a
\]
\[
\Rightarrow X'' - 2X' = aX \quad \text{and} \quad Y' = aY
\]
\[
\Rightarrow X'' - 2X' - aX = 0 \quad \text{and} \quad Y' - aY = 0
\]

**Case 1** Consider the ordinary differential equation $X'' - 2X' - aX = 0$ and the auxiliary equation is $m^2 - 2m - a = 0$ where
\[
m = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = 1 \pm \sqrt{1 + a}
\]
\[
\Rightarrow X = c_1e^{(1+\sqrt{1+a})x} + c_2e^{(1-\sqrt{1+a})x}
\]

Note :- If the roots of an auxiliary equations are real and distinct (say $m_1 \text{ and } m_2$), the solution is $y = c_1e^{m_1x} + c_2e^{m_2x}$

**Case 2** Consider the ordinary differential equation $Y' + aY = 0$ and
the auxiliary equation is \( m + a = 0 \) and \( m = -a \).

\[ y = c_3 e^{-ay} \]

The required solution is \( z = XY \)

\[ z = \left( c_1 e^{(1+\sqrt{1+a})x} + c_2 e^{(1-\sqrt{1+a})x} \right) e^{-ay} \]

\[ z = \left( c_1 c_3 e^{(1+\sqrt{1+a})x} + c_2 c_3 e^{(1-\sqrt{1+a})x} \right) e^{-ay} \]

\[ z = \left( k_1 e^{(1+\sqrt{1+a})x} + k_2 e^{(1-\sqrt{1+a})x} \right) e^{-ay} \]

where \( c_1 c_2 = k_1 \) and \( c_2 c_3 = k_2 \)

**Example 25.** Using the method of separation of variables, solve \( \frac{\partial u}{\partial x} = 2 \frac{\partial u}{\partial t} + u \) where \( u(x,0) = 6e^{-3x} \).

Solution:-

\[ \frac{\partial u}{\partial x} = 2 \frac{\partial u}{\partial t} + u \]

Assume the solution of given Partial differential equation is \( u = XT \)

where \( X \) is a function of \( x \) alone and \( T \) that of \( t \) alone.

\[ \frac{\partial u}{\partial x} = X'T, \quad \frac{\partial u}{\partial t} = XT' \]

Substitute these values in given PDE

\[ X'T = 2XT' + XT \]

\[ X'T - XT = 2XT' \]

\[ (X' - X)T = 2XT' \]

\[ \frac{X' - X}{2X} = \frac{2XT'}{T} \]

Since \( x \) and \( y \) are independent variables, it can only be true if each side is equal to the same constant, \( a \) (say)

\[ \frac{X' - X}{2X} = a \text{ and } \frac{T'}{T} = a \]

\[ X' - X = 2aX \text{ and } T' - aT = 0 \]

\[ X' - 2aX = 0 \text{ and } T' - aT = 0 \]

\[ X' - (2a + 1)X = 0 \text{ and } T' - aT = 0 \]

**Case 1** Consider the ordinary differential equation \( X' - (2a + 1)X = 0 \)
and the auxiliary equation is \( m - (2a + 1) = 0 \Rightarrow m = 2a + 1 \)
\[ \Rightarrow X = c_1 e^{(2a+1)x} \]

**Case 2** Consider the ordinary differential equation \( T' - aT = 0 \) and the auxiliary equation is \( m - a = 0 \Rightarrow m = a \)
\[ \Rightarrow T = c_2 e^{at} \]

The required solution is \( u = XT \Rightarrow u = c_1 e^{(2a+1)x} c_2 e^{at} \)
\[ \Rightarrow u = c_1 c_2 e^{(2a+1)x} e^{at} \Rightarrow u = k e^{(2a+1)x} e^{at} \text{ where } c_1 c_2 = k \]

Given that \( u(x, 0) = 6e^{-3x} \) [i.e. at \( t = 0 \), \( u = 6e^{-3x} \)]
\[ \Rightarrow 6e^{-3x} = ke^{(2a+1)x} \Rightarrow k = 6 \text{ and } 2a + 1 = -3 \Rightarrow a = -2 \]
\[ \Rightarrow \text{The required solution is } u = 6e^{-3x} e^{-2t} \Rightarrow u = 6e^{-(3x+2t)} \]

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**References**


MAT 201

RAJAGIRI SCHOOL OF ENGINEERING & TECHNOLOGY
DEPARTMENT OF MATHEMATICS
TUTORIAL / ASSIGNMENT/ UNIT-WISE QUESTION BANK RECORD BOOK
COURSE:- MAT 201; PARTIAL DIFFERENTIAL EQUATIONS AND COMPLEX ANALYSIS
Branch: COMMON FOR ALL BRANCHES (EXCEPT CS & IT)

MODULE II APPLICATION OF PARTIAL DIFFERENTIAL EQUATIONS

TUTORIAL

1. Write down the assumptions involved in deriving the one dimensional wave equation.

2. Solve using the method of separation of variables.

\[ \frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \quad \text{for } 0 < x < 2, \ t > 0 \]

\[ y(0, t) = y(2, t) = 0 \]

\[ y(x, 0) = 0, \ \frac{\partial y}{\partial t}(0, t) = \begin{cases} 2x & 0 \leq x \leq 1 \\ 0 & 1 < x \leq 2 \end{cases} \]

3. Write any two assumptions involved in deriving one dimensional heat equations.

4. A rod of length \( l \) is heated in such a way that its ends \( A \) and \( B \) are at zero temperature. If initially its temperature is given by \( f(x) = \frac{cx(l-x)}{l^2} \), \( 0 \leq x \leq l \), find the temperature at time \( t \).

5. Find the temperature \( u(x, t) \) in a bar which is perfectly insulated laterally, whose ends are kept at temperature \( 0^\circ \) and whose initial temperature is \( f(x) = x(10 - x) \) given that its length is 10 cm, constant cross section area 1 cm\(^2\), density 10.5 gm/cm\(^2\), thermal conductivity 1.04 cal/cm deg sec and specific heat 0.056 cal/gm deg

ASSIGNMENT

1. If a string of length \( l \) is initially at rest in equilibrium and each of its points is given a velocity \( v = \begin{cases} cx & 0 < x \leq l/2 \\ c(l-x) & l/2 < x < l \end{cases} \), determine the displacement \( y(x, t) \).

2. Solve using separation of variables \[ \frac{\partial^2 y}{\partial t^2} = 9 \frac{\partial^2 y}{\partial x^2} \] subject to \( y(0, t) = y(4, t) = 0, \ y(x, 0) = 2 \sin(\pi x), \ \frac{\partial y}{\partial t}(x, t) = \]

3. Solve completely the equation \[ \frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \] representing the vibrations of a string of length \( l \), fixed at both ends, given that \( y(0, t) = y(l, t) = 0, \ y(x, 0) = f(x), \ \left( \frac{\partial y}{\partial t} \right)_{t=0} = 0 \)

4. Obtain the general solution of the wave equation using the method of separation of variables.

5. Using d’Alembert’s method, find the deflection of a vibrating string of unit length having fixed ends, with 0 initial velocity and initial deflection

(a) \( f(x) = a(x - x^2) \)
(b) \( f(x) = e \sin^2(\pi x) \)

6. Find the temperature distribution in a rod of length 2\( m \) whose end points are maintained temperature zero and the initial temperature \( f(x) = 100(2x - x^3) \).

7. Write the mathematically possible solution of one dimensional heat equation.

8. In the heat equation \( \frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \) what does \( c^2 \) indicate. State the boundary conditions and initial conditions for solving it.

9. Find the temperature in a bar of length 10\( cm \) whose preferable insulated laterally whose ends are kept at zero temperature and whose initial temperature is

\[
 f(x) = \begin{cases} 
 x, & 0 \leq x \leq 5 \\
 10 - x, & 5 \leq x \leq 10 
\end{cases}
\]

10. Find the temperature distribution in a rod of length \( l \) whose end points are maintained temperature zero and the initial temperature in the rod is \( \sin \left( \frac{\pi x}{l} \right) \).

**UNIT-WISE QUESTION BANK**

1. Derive the one dimensional wave equation.

2. Find the solution of the wave equation corresponding to the initial deflection

\[
f(x) = \begin{cases} 
 \frac{2k}{l} x, & 0 < x < l/2 \\
 \frac{2k}{l} (l - x), & l/2 < x < l 
\end{cases}
\]

and initial velocity 0.

3. A tightly stretched string of length \( l \) has its ends fastened at \( x = 0, x = l \). The mid point of the string is then taken to a height \( h \) and the string is then released from rest in that position. Find the lateral displacement at a point of the string at time \( t \) from the instant of release.

4. Solve \( \frac{\partial^2 y}{\partial t^2} = 4 \frac{\partial^2 y}{\partial x^2} \) using the method of separation of variables subject to

\[
y(0, t) = y(5, t) = 0
\]

\[
y(x, 0) = 0, \quad \frac{\partial y}{\partial t} = \begin{cases} 
 0, & 0 \leq x < 4 \\
 5 - x, & 4 \leq x \leq 5 
\end{cases}
\]

5. Solve \( \frac{\partial^2 y}{\partial t^2} = 8 \frac{\partial^2 y}{\partial x^2} \) using the method of separation of variables subject to

\[
y(0, t) = y(2x, t) = 0
\]

\[
y(x, 0) = \begin{cases} 
 3x, & 0 \leq x \leq \pi \\
 5\pi - 3x, & \pi < x \leq 2\pi 
\end{cases}, \quad \frac{\partial y}{\partial t} = 0
\]

6. A tightly stretched string with fixed end points at \( x = 0, x = l \) is initially in the position

\[
f(x) = \begin{cases} 
 x, & 0 \leq x \leq 1/2 \\
 1 - x, & 1/2 \leq x \leq 1 
\end{cases}
\]

If it is released from this position with a velocity \( v \) perpendicular to the \( x \) axis, find the displacement \( y(x, t) \) at any point \( x \) of the string at any time \( t > 0 \).
7. The points of trisection of a string are pulled aside through the same distance on opposite sides of the equilibrium position and the string is released from rest. Derive an expression for the displacement of the string at subsequent times and show that the midpoint of the string always remains at rest.

8. Solve the wave equation \( \frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \) under the conditions

\[
y(0, t) = y(t, t) = 0
\]

\[
y(x, 0) = f(x) \quad \text{and} \quad \left( \frac{\partial y}{\partial t} \right)_{t=0} = g(x)
\]

9. Solve one dimensional heat equation \( \frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \) subject to the conditions \( u(x, t) \neq \infty \) if \( t \to \infty \): \( u(0, t) = 0 = u(\pi, t) : u(x, 0) = \pi - x^2, 0 \leq x \leq \pi \).

10. Derive one dimensional heat equation

11. Find the temperature in a laterally insulated bar of length \( L \) whose ends are kept at temperature zero if the initial temperature is \( f(x) = \begin{cases} x, & 0 < x < \frac{L}{2} \\ L - x, & \frac{L}{2} < x < L \end{cases} \)

12. Find the temperature in a bar of length 2m whose ends are kept at zero temperature and lateral surface insulated if the initial temperature is \( u(x, 0) = \sin \frac{\pi x}{2} + 3 \sin \frac{5\pi x}{2} \)

13. Solve the boundary value problem \( \frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \) with the conditions \( \frac{\partial u}{\partial t} (0, t) = 0, \ u(t, t) = 0, \forall t \geq 0 \) and \( u(x, 0) = 20x \)

14. Solve the boundary value problem \( \frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \) with the conditions \( u(0, t) = 0, \ u(t, 0) = 0, \forall 0 \leq x \leq 1 \) and \( u(x, 0) = 3 \sin n\pi x \forall t \geq 0 \)

15. A homogeneous rod of conducting material of length 100 cm has its ends kept at zero temperature and the temperature initially is \( u(x, 0) = \begin{cases} x, & 0 \leq x \leq 50 \\ 100 - x, & 50 < x < 100 \end{cases} \). Find the temperature \( u(x, t) \) at any time.
Course Handout

MAT 201 : Partial Differential Equations and Complex Analysis
Module 2 : Applications of Partial Differential Equations

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Pre requisites

(i) Method of solution of linear ODEs
(ii) Fourier series
(iii) Half Range expansions
1 Applications of PDEs

Partial differential equations are seen in a lot of problems in physics and engineering. In this module, we look at two special PDEs - the wave and heat equations. We shall see another well known PDE, the Laplace’s equation in Module 3.

In this module, we will derive the heat and wave equations and solve them.

2 The Wave Equation

The wave equation is a second order PDE which is used to describe the small oscillations at fixed speed of some quantity $y$.

It is given by

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$$

where $c$ is the speed of the wave.

For example, if a stretched string (as in a guitar or a violin) is plucked, the string oscillates about equilibrium and $y$ would be the vertical displacement of the string.

Or, if a tightly stretched membrane (as in a drum) is hit, it too will oscillate about equilibrium, with $y$ being displacement.

The wave equation describes many phenomena, like gravitational waves, light waves, sound waves and oscillations of strings.

2.1 Deriving the wave equation

Start with an elastic string of length $l$ which is tightly stretched between 2 pegs as in a guitar string. We want to know how the string moves if we give it a small displacement (plucking the guitar string) and then leave it to vibrate.

Place the string on the $x$ - axis from $0$ to $l$, with the ends fixed at $x = 0, x = l$.

We want to find a function $y(x, t)$ such that at any fixed time $t_0 > 0$, the graph of $y(x, t_0)$ gives the shape of the string at time $t_0$.

This $y(x, t)$ is called the position function for the string and we want to derive the PDE satisfied by it.

The string is assumed to have a constant mass per unit length $\rho$ and subjected to a constant tension $T$. 
To simplify the calculations, make the following assumptions:

(i) There are no external forces acting on the string - neglect damping forces like air resistance.

(ii) The tension $T$ is considered to be very large as compared to the weight of the string. It is also assumed that $T$ always acts tangentially to the string.

(iii) Individual particles of the string move only vertically, so that the string vibrates in the $xy$ plane.

Consider the following figure:

Line segment $AB$ where $A = (0,0), B = (l,0)$ is the equilibrium position of the string. The string is fixed at $A$ and $B$, so these points do not move.

This string is set vibrating. At time $t$, let the string be in the position shown in the figure.

Take a small element $PQ$ of the string, between the points $P(x, y)$ and $Q(x + \delta x, y + \delta y)$.

The tension $T$ acts tangentially, and the tangents at $P, Q$ make angles $\psi, \psi + \delta \psi$ respectively with respect to the $x$ axis.

Apply Newton’s second law:

\[
\text{Net force on } PQ \text{ due to tension} = \text{mass } \times \text{ acceleration of } PQ
\]

\[
F = m \ddot{a}
\]

On the right hand side, mass of $PQ$ is the mass per unit length times the length of $PQ$. Since $PQ$ is a small element, we can consider the length of $PQ$ to be $\delta x$. 

![Diagram of string vibration](image-url)
So \( m = \rho \times \delta x \).

As the string moves only vertically, the horizontal acceleration \( \frac{\partial^2 x}{\partial t^2} = 0 \).

Therefore \( PQ \) is moving with acceleration \( \frac{\partial^2 y}{\partial t^2} \).

This leads to the equation

\[
F = \rho \delta x \frac{\partial^2 y}{\partial t^2}
\]

Now evaluate the force due to tension. Again, we look at the vertical component of the force.

\[
F = T \sin(\psi + \delta \psi) - T \sin(\psi) = T[\sin(\psi + \delta \psi) - \sin(\psi)]
\]

Since the angles are small, \( \sin \psi \approx \tan \psi \), so

\[
F = T[\tan(\psi + \delta \psi) - \tan(\psi)]
\]

Remember that \( \tan \psi \) is the slope of the tangent at \( P(x, y) \), and that slope of the tangent is also given by \( \frac{\partial y}{\partial x} \).

However \( y \) here is a function of \( x, t \), so

\[
\tan(\psi) = \frac{\partial y}{\partial x}(x, t)
\]

\[
\tan(\psi + \delta \psi) = \frac{\partial y}{\partial x}(x + \delta x, t)
\]

Substituting,

\[
F = T \frac{\partial y}{\partial x}(x + \delta x, t) - \frac{\partial y}{\partial x}(x, t) = \rho \delta x \frac{\partial^2 y}{\partial t^2}
\]

Rearranging the above equation, we find

\[
\frac{\partial^2 y}{\partial t^2} = \frac{T}{\rho} \left( \frac{\partial y}{\partial x}(x + \delta x, t) - \frac{\partial y}{\partial x}(x, t) \right) \delta x
\]

**Result:**

\[
\frac{\partial y}{\partial x} = \lim_{\delta x \to 0} \frac{y(x + \delta x, t) - y(x, t)}{\delta x}
\]

Taking the limit \( Q \to P \), \( \delta x \to 0 \), so

\[
\frac{\partial^2 y}{\partial t^2} = \frac{T}{\rho} \frac{\partial^2 y}{\partial x^2}
\]
Let \( \frac{\rho}{\rho} = c^2 \), so the equation is

\[
\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}
\]

This is the PDE describing the transverse vibrations of a string, and it is called the one dimensional wave equation.

## 3 Solving the wave equation

We will learn 2 methods for solving the wave equation:

(i) Method of separation of variables

(ii) d’Alembert’s solution

### 3.1 Method of separation of variables

The method of separation of variables discussed in this section gives a Fourier series solution to the wave equation.

We have the wave equation

\[
\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}
\]

Assume that the solution \( y(x, t) \) is of the form \( y(x, t) = X(x)T(t) \), where \( X \) is a function of \( x \) and \( T \) is a function of \( t \) alone.

If we evaluate the partial derivatives of \( y \),

\[
\frac{\partial^2 y}{\partial x^2} = X''(x)T(t)
\]

\[
\frac{\partial^2 y}{\partial t^2} = X(x)T''(t)
\]

Substitute for the derivatives in the wave equation:

\[
X(x)T''(t) = c^2 X''(x)T(t)
\]

\[
\Rightarrow \frac{X''(x)}{X(x)} = \frac{1}{c^2} \frac{T''(t)}{T(t)}
\]
Let \( \frac{T}{\rho} = c^2 \), so the equation is

\[ \frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \]

This is the PDE describing the transverse vibrations of a string, and it is called the one-dimensional wave equation.

### 3 Solving the wave equation

We will learn 2 methods for solving the wave equation:

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#### 3.1 Method of separation of variables

The method of separation of variables discussed in this section gives a Fourier series solution to the wave equation.

We have the wave equation

\[ \frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \]

Assume that the solution \( y(x, t) \) is of the form \( y(x, t) = X(x)T(t) \), where \( X \) is a function of \( x \) and \( T \) is a function of \( t \) alone.

If we evaluate the partial derivatives of \( y \),

\[ \frac{\partial^2 y}{\partial x^2} = X''(x)T(t) \]

\[ \frac{\partial^2 y}{\partial t^2} = X(x)T''(t) \]

Substitute for the derivatives in the wave equation:

\[ X(x)T''(t) = c^2 X''(x)T(t) \]

\[ \Rightarrow \frac{X''(x)}{X(x)} = \frac{T''(t)}{c^2 T(t)} \]
The left hand side of this equation is a function of \(x\) only and the right hand side of \(t\) only. So this equation will hold only if each side of this equation is a constant. Call this constant \(k\), so that

\[
\frac{X''(x)}{X(x)} = k, \quad \frac{T''(t)}{c^2 T(t)} = k
\]

This gives us 2 ordinary differential equations for \(X(x)\) and \(T(t)\):

\[
X'' - kX = 0, \quad T'' - kc^2 T = 0
\]

Both these ODEs are homogeneous second order linear ODEs, so we can solve them using the roots of the corresponding auxiliary equations.

The auxiliary equation for the first ODE is \(m^2 - k = 0\), and for the second equation is \((m')^2 - kc^2 = 0\).

The solutions of the auxiliary equations depend on the value of \(k\). So there are 3 possibilities.

(i) \(k > 0\):

If \(k > 0\), we can write \(k = p^2\), for some \(p\).

Then the auxiliary equations are \(m^2 = p^2\), and \((m')^2 = c^2 p^2\).

The roots of the auxiliary equations are \(m = \pm p, \ m' = \pm cp\) respectively.

Then, the solutions of the ODEs are \(X(x) = c_1 e^{px} + c_2 e^{-px}, T(t) = c_3 e^{ct} + c_4 e^{-ct}\).

So the solution of the wave equation in this case is

\[
y(x, t) = (c_1 e^{px} + c_2 e^{-px})(c_3 e^{ct} + c_4 e^{-ct})
\]

(ii) \(k < 0\):

If \(k < 0\), we can write \(k = -p^2\).

This gives the new auxiliary equations \(m^2 = -p^2\), \((m')^2 = -c^2 p^2\), with the roots \(m = \pm ip, \ m' = \pm icp\) where \(i = \sqrt{-1}\).

The solutions of the ODEs for \(k < 0\) are \(X(x) = c_5 \cos(px) + c_6 \sin(px), T(t) = c_7 \cos(cpt) + c_8 \sin(cpt)\).

So when \(k < 0\), the solution is
\[ y(x, t) = (c_1 \cos(px) + c_6 \sin(px))(c_7 \cos(cpt) + c_8 \sin(cpt)) \]

(iii) \(k = 0\)

In this last case, the auxiliary equations become \(m^2 = 0, \quad (m')^2 = 0\). The roots are \(m = 0, 0\) and \(m' = 0, 0\).

Both auxiliary equations have repeated roots and the solutions will be \(X(x) = c_0 + c_1 x, T(t) = c_{11} + c_{12} t\).

For \(k = 0\), the solution is

\[ y(x, t) = (c_9 + c_{13} x)(c_{11} + c_{12} t) \]

We found 3 possible solutions for the wave equation but \(y(x, t)\) is the position function for a vibrating string, and so should be periodic.

Therefore, out of the 3 possible solutions, we choose the one that 

is periodic i.e. the solution involving trigonometric terms.

**By the method of separation of variables, the general solution of the wave equation is**

\[ y(x, t) = (c_1 \cos(px) + c_2 \sin(px))(c_3 \cos(cpt) + c_4 \sin(cpt)) \]

The solution of the wave equation is given above, but to evaluate the position function for any specific situation, we need additional information.

This is usually given by the initial conditions at time \(t = 0\), namely the initial displacement \(y(x, 0)\) and the initial velocity \(\frac{\partial y}{\partial t}(x, 0)\).

Also the end points of the string of length \(l\) are fixed at all times and so have displacement 0. This gives the boundary conditions for the problem \(y(0, t) = 0, y(l, t) = 0\).

**Example 1. Vibrating string with given initial displacement and 0 initial velocity:**

A string is stretched and fastened to 2 points at a distance \(l\) from each other. Motion is started by displacing the string in the form \(y = a \sin \left( \frac{\pi x}{l} \right) \) from which it is released at time \(t = 0\).

Show that the displacement of any point at a distance \(x\) from one end at time \(t\) is given by \(y(x, t) = a \sin \left( \frac{\pi x}{l} \right) \cos \left( \frac{\pi ct}{l} \right) \).
Solution:

The displacement satisfies the wave equation

\[ \frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \]

subject to the boundary conditions

\[ y(0, t) = 0, y(l, t) = 0 \]

and the initial conditions are given by the initial displacement and initial velocity

\[ y(x, 0) = a \sin \left( \frac{\pi x}{l} \right), \quad \frac{\partial y}{\partial t}(x, 0) = 0 \]

To find the displacement, we have to find the general solution of the PDE and then apply the initial and boundary conditions to find the value of the arbitrary constants.

We already found the solution of the wave equation for a vibrating string as

\[ y(x, t) = (c_1 \cos(p x) + c_2 \sin(p x))(c_3 \cos(c p t) + c_4 \sin(c p t)) \]

First we apply the boundary conditions:

\[ y(0, t) = (c_1 \cos 0 + c_2 \sin 0)(c_3 \cos(c p t) + c_4 \sin(c p t)) = c_1 (c_3 \cos(c p t) + c_4 \sin(c p t)) = 0 \]

For the above relation to be true for all \( t \), \( c_1 \) has to be 0.

So our solution becomes

\[ y(x, t) = c_2 \sin(p x)(c_3 \cos(c p t) + c_4 \sin(c p t)) \]

Applying the second boundary condition:

\[ y(l, t) = c_2 \sin(pl)(c_3 \cos(c p t) + c_4 \sin(c p t)) = 0 \]

Again, for this statement to hold for all values of \( t \), we require \( \sin(pl) = 0 \). Solving this equation gives \( pl = n\pi \), or \( p = \frac{n\pi}{l} \), where \( n \) is an integer.

Substituting for \( p \):

\[ y(x, t) = c_2 \sin\left( \frac{n\pi}{l} x \right)(c_3 \cos\left( \frac{n\pi}{l} ct \right) + c_4 \sin\left( \frac{n\pi}{l} ct \right)) \]
And
\[ \frac{\partial y}{\partial t} = c_2 \sin\left(\frac{n\pi}{l}x\right) \frac{nc\pi}{l} \left(-c_3 \sin\left(\frac{n\pi}{l}ct\right) + c_4 \cos\left(\frac{n\pi}{l}ct\right)\right) \]

Now we can apply the second initial condition:
\[ \frac{\partial y}{\partial t}(x, 0) = c_2 \sin\left(\frac{n\pi}{l}x\right) \frac{nc\pi}{l} \left(-c_3 \sin 0 + c_4 \cos 0\right) = c_2 c_4 \frac{nc\pi}{l} = 0 \]

By the above, either \( c_2 = 0 \) or \( c_4 = 0 \). But if \( c_2 = 0 \), the solution becomes \( y(x, t) = 0 \), the trivial solution.
So \( c_4 = 0 \), and the solution is
\[ y(x, t) = c_2 c_3 \sin\left(\frac{n\pi}{l}x\right) \cos\left(\frac{n\pi}{l}ct\right) \]

The above function is a solution of the PDE for any integer value of \( n \). So the sum of the sum of all such functions is also a solution.

Denote \( c_2 c_3 \) by \( b_n \). Then the new solution found by adding all of the above is
\[ y(x, t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{l}x\right) \cos\left(\frac{n\pi}{l}ct\right) \]

Finally, apply the remaining initial condition:
\[ y(x, 0) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{l}x\right) \cos 0 = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{l}x\right) = a \sin\left(\frac{\pi x}{l}\right) \]

Expanding,
\[ b_1 \sin\left(\frac{\pi}{l}x\right) + b_2 \sin\left(\frac{2\pi}{l}x\right) + b_3 \sin\left(\frac{3\pi}{l}x\right) + \cdots = a \sin\left(\frac{\pi x}{l}\right) \]

By comparison, \( b_1 = a \), \( b_2 = b_3 = \cdots = 0 \). Now we can substitute these constants to get the solution
\[ y(x, t) = a \sin\left(\frac{\pi x}{l}\right) \cos\left(\frac{n\pi ct}{l}\right) \]

**Example 2. Vibrating string with given initial displacement and 0 initial velocity:**

A tightly stretched string with fixed end points \( x = 0 \) and \( x = l \) is initially in the position \( y = y_0 \sin^2\left(\frac{\pi x}{l}\right) \). If it is released from rest from this position, find the displacement \( y(x, t) \).
Solution:

The displacement satisfies the wave equation

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$$

subject to the boundary conditions

$$y(0, t) = 0, y(l, t) = 0$$

and the initial conditions

$$y(x, 0) = y_0 \sin \left( \frac{\pi x}{l} \right), \quad \frac{\partial y}{\partial t}(x, 0) = 0$$

As before, we find the general solution of the PDE and then substitute the initial and boundary conditions.

However, notice that the boundary conditions and the initial velocity are the same as the previous problem.

So substituting the boundary conditions and the initial velocity gives the same solution function, namely

$$y(x, t) = \sum_{n=1}^{\infty} b_n \sin \left( \frac{n\pi x}{l} \right) \cos \left( \frac{n\pi}{l} ct \right)$$

The last thing to do is substitute the remaining initial condition:

$$y(x, 0) = y_0 \sin \left( \frac{\pi x}{l} \right) = \sum_{n=1}^{\infty} b_n \sin \left( \frac{n\pi x}{l} \right)$$

To compare both sides of the equation, we rewrite the left hand side.

**Result:**

$$\sin^3 \theta = \frac{3}{4} \sin \theta - \frac{1}{4} \sin 3\theta$$

So

$$\frac{y_0}{4} \left[ 3 \sin \left( \frac{\pi x}{l} \right) - \sin \left( \frac{3\pi x}{l} \right) \right] = \sum_{n=1}^{\infty} b_n \sin \left( \frac{n\pi x}{l} \right)$$

$$\frac{y_0}{4} \left[ 3 \sin \left( \frac{\pi x}{l} \right) - \sin \left( \frac{3\pi x}{l} \right) \right] = b_1 \sin \left( \frac{\pi}{l} x \right) + b_2 \sin \left( \frac{2\pi}{l} x \right) + b_3 \sin \left( \frac{3\pi}{l} x \right) + \cdots$$

Comparing the coefficients, \(b_1 = \frac{3y_0}{4}, b_2 = 0, b_3 = \frac{-y_0}{4}, b_4 = b_5 = \cdots = 0\).
Substitute the values of the coefficients to find the solution as

$$y(x, t) = \frac{3y_0}{4} \sin \left( \frac{\pi}{l} x \right) \cos \left( \frac{\pi c}{l} t \right) - \frac{y_0}{4} \sin \left( \frac{3\pi}{l} x \right) \cos \left( \frac{3\pi c}{l} t \right)$$

Example 3. Vibrating string with given initial displacement and 0 initial velocity:

A tightly stretched flexible string has its ends fixed at $x = 0$ and $x = l$. At $t = 0$, the string is given a shape defined by $f(x) = \mu x (l - x)$ where $\mu$ is a constant and then released.

Find the displacement of any point $x$ of the string at any time $t > 0$.

Solution:

The displacement satisfies the wave equation

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}$$

subject to the boundary conditions

$$y(0, t) = 0, \ y(l, t) = 0$$

and the initial conditions

$$y(x, 0) = \mu x (l - x), \ \frac{\partial y}{\partial t}(x, 0) = 0$$

Again, notice that the boundary conditions and the initial velocity are the same as the previous 2 problems.

So substituting the boundary conditions and the initial velocity gives the same solution function, namely

$$y(x, t) = \Sigma_{n=1}^{\infty} b_n \sin \left( \frac{n\pi}{l} x \right) \cos \left( \frac{n\pi c}{l} t \right)$$

All that remains to be done is to apply the remaining initial condition and then find the coefficients $b_n$.

Substituting the initial condition:

$$y(x, 0) = \mu x (l - x) = \mu (lx - x^2) = \Sigma_{n=1}^{\infty} b_n \sin \left( \frac{n\pi}{l} x \right)$$
Unlike the previous 2 questions, here we have no way of directly comparing the coefficients on both sides.

But the expression on the right hand side is the **half range sine series** of some function defined on \((0, l)\).

### Result:
The half range sine series of a function \(f(x)\) defined on \(0 < x < l\) is given by
\[
\sum_{n=1}^{\infty} b_n \sin \left( \frac{n \pi x}{l} \right)
\]
where
\[
b_n = \frac{2}{l} \int_{0}^{l} f(x) \sin \left( \frac{n \pi x}{l} \right) dx
\]
Also \(f(x) = \sum_{n=1}^{\infty} b_n \sin \left( \frac{n \pi x}{l} \right)\) in \(0 < x < l\).

By the above result, \(b_n\) are the coefficients of the half range sine series of \(\mu(lx - x^2)\).

Again by the above result,
\[
b_n = \frac{2}{l} \int_{0}^{l} \mu(lx - x^2) \sin \left( \frac{n \pi x}{l} \right) dx
\]

### Result: Bernoulli’s Rule
\[
\int uv = u_1v_1 + u_2v_2 + \cdots
\]
where
\[
u_1 = u', u_2 = u_1', \cdots, v_1 = \int u, v_2 = \int v_1, \cdots
\]

Apply Bernoulli’s rule to find \(b_n\).

\[
b_n = \frac{2\mu}{l} \left[ (lx - x^2) - \cos \left( \frac{n \pi x}{l} \right) \right] - \left[(l - 2x) - \sin \left( \frac{n \pi x}{l} \right) \right]
\]
\[
+ \left[(-2) \cos \left( \frac{n \pi x}{l} \right) \right]_{0}^{l}
\]

The first 2 terms go to 0 when the limits are applied, and we are left with
\[
b_n = \frac{4\mu l^2}{l^3} \left[1 - \cos \pi x\right] = \frac{4\mu l^2}{l^3} \left[1 - (-1)^n\right]
\]

Therefore
\[
b_n = \begin{cases} 
0 & \text{n is even i.e. } n = 2m \\
\frac{4\mu l^2}{l^3} & \text{n is odd i.e. } n = 2m - 1
\end{cases}
\]

Finally, after substituting for the constants, we get the solution
\[
y(x, t) = \frac{8\mu l^2}{\pi^3} \sum_{m=1}^{\infty} \frac{1}{(2m - 1)^3} \sin \left( \frac{(2m - 1)\pi x}{l} \right) \cos \left( \frac{(2m - 1)\pi c}{l} t \right)
\]
Example 4. Vibrating string with given initial velocity and 0 initial displacement: A tightly stretched string of length \( l \) with fixed ends is initially in equilibrium position. It is set vibrating by giving each point a velocity \( v_0 \sin \left( \frac{n\pi}{l} x \right) \). Find the displacement \( y(x, t) \).

Solution:

The displacement satisfies the wave equation

\[
\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2}
\]

subject to the boundary conditions

\[ y(0, t) = 0, \ y(l, t) = 0 \]

and the initial conditions

\[ y(x, 0) = 0, \ \frac{\partial y}{\partial t}(x, 0) = v_0 \sin \left( \frac{n\pi}{l} x \right) \]

In this question, the PDE and boundary conditions haven’t changed, but the initial velocity is now non-zero, and there is no initial displacement.

We had derived the general solution for the wave equation at the beginning. From the first example, we know what the solution is after applying the boundary conditions.

\[ y(x, t) = c_2 \sin \left( \frac{n\pi}{l} x \right) \left( c_3 \cos \left( \frac{n\pi}{l} ct \right) + c_4 \sin \left( \frac{n\pi}{l} ct \right) \right) \]

Now apply the first initial condition:

\[ y(x, 0) = c_2 c_3 \sin \left( \frac{n\pi x}{l} \right) \]

For this condition to be true for all \( x \), \( c_2 c_3 = 0 \). Then either \( c_2 = 0 \) or \( c_3 = 0 \). But \( c_2 = 0 \) leads to the trivial solution \( y = 0 \), so \( c_3 \) should be 0.

So the solution is of the form

\[ y(x, t) = c_2 c_4 \sin \left( \frac{n\pi x}{l} \right) \sin \left( \frac{n\pi}{l} ct \right) \]

Let \( c_2 c_4 = b_n \).

The above function is a solution for all integer values of \( n \). Hence the sum of such solutions for all values of \( n \) is also a solution.
This new solution is

\[ y(x, t) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{l}x\right) \sin\left(\frac{n\pi}{l}ct\right) \]

Apply the remaining initial condition:

\[ \frac{\partial y}{\partial t} = \sum_{n=1}^{\infty} b_n \frac{cn\pi}{l} \sin\left(\frac{n\pi}{l}x\right) \cos\left(\frac{n\pi}{l}ct\right) \]

\[ \frac{\partial y}{\partial t}(x, 0) = v_0 \sin\left(\frac{\pi x}{l}\right) = \sum_{n=1}^{\infty} b_n \frac{cn\pi}{l} \sin\left(\frac{n\pi}{l}x\right) \]

Applying trigonometric identities,

\[ \frac{v_0}{4} \left[ 3 \sin\left(\frac{\pi x}{l}\right) - \sin\left(\frac{3\pi x}{l}\right) \right] = \sum_{n=1}^{\infty} b_n \frac{cn\pi}{l} \sin\left(\frac{n\pi}{l}x\right) \]

Expand the right hand side and equate coefficients.

\[ \frac{v_0}{4} \left[ 3 \sin\left(\frac{\pi x}{l}\right) - \sin\left(\frac{3\pi x}{l}\right) \right] = b_1 \frac{c\pi}{l} \sin\left(\frac{\pi x}{l}\right) + b_2 \frac{2c\pi}{l} \sin\left(\frac{2\pi x}{l}\right) + b_3 \frac{3c\pi}{l} \sin\left(\frac{3\pi x}{l}\right) + \cdots \]

\[ \frac{c\pi}{l} b_1 = \frac{3v_0}{4}, \quad \frac{2c\pi}{l} b_2 = 0, \quad \frac{3c\pi}{l} b_3 = -\frac{v_0}{4}, \quad b_n \frac{cn\pi}{l} = 0 \quad \text{when} \quad n \geq 4 \]

This gives the coefficient values

\[ b_1 = \frac{3v_0l}{4c\pi}, \quad b_2 = b_3 = b_5 = \cdots = 0 \]

The final solution is

\[ y(x, t) = \frac{3v_0l}{4c\pi} \sin\left(\frac{\pi x}{l}\right) \sin\left(\frac{ct}{l}\right) - \frac{v_0l}{12c\pi} \sin\left(\frac{3\pi x}{l}\right) \sin\left(\frac{3\pi t}{l}\right) \]

Example 5. Vibrating string with given initial velocity and 0 initial displacement: A tightly stretched string with fixed end points \( x = 0, \ x = l \) is initially at rest in its equilibrium position. If it is set vibrating by giving to each of its points a velocity \( \lambda \Delta(l - x) \), find the displacement of the string at any distance \( x \) from one end at any time \( t \).

Solution:

The displacement satisfies the wave equation
\[ \frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \]

subject to the boundary conditions

\[ y(0, t) = 0, y(l, t) = 0 \]

and the initial conditions

\[ y(x, 0) = 0, \frac{\partial y}{\partial t} (x, 0) = \lambda x (l - x) \]

Observe that the boundary conditions and the initial displacement are the same as the previous problem.

So substituting the boundary conditions and the initial displacement gives the same solution function, namely

\[ y(x, t) = \Sigma_{n=1}^{\infty} b_n \sin\left(\frac{n\pi}{l} x\right) \sin\left(\frac{n\pi}{l} ct\right) \]

Now apply the remaining initial condition

\[ \frac{\partial y}{\partial t} = \Sigma_{n=1}^{\infty} c t \pi b_n \sin\left(\frac{n\pi}{l} x\right) \cos\left(\frac{n\pi}{l} ct\right) \]

At \( t = 0 \)

\[ \frac{\partial y}{\partial t} (x, 0) = \lambda (lx - x^2) = \frac{c t \pi}{l} \Sigma_{n=1}^{\infty} n b_n \sin\left(\frac{n\pi}{l} x\right) \]

The right hand side is the half range sine series of \( \lambda (lx - x^2) \). Using the formula for the coefficients of the half range sine series gives

\[ \frac{c t \pi}{l} n b_n = \frac{2}{l} \int_0^l \lambda (lx - x^2) \sin\left(\frac{n\pi}{l} x\right) dx \]

Apply Bernoulli’s rule:

\[ = \frac{2\lambda}{l} \left[ lx - x^2 \right] - \left( \frac{\text{max}}{l} \right) - \left( i - 2x \right) \sin\left( \frac{\text{max}}{l} \right) + \left( -2 \right) \cos\left( \frac{\text{max}}{l} \right) \]

The first 2 terms cancel out after substituting the limits, so

\[ \frac{c t \pi}{l} n b_n = \frac{4\lambda^2}{n^2 \pi^2} \left[ 1 - \cos n \pi \right] = \frac{4\lambda^2}{n^2 \pi^2} \left[ 1 - (-1)^n \right] \]
\[ b_n = \frac{4LM^3}{cn \pi^3} [1 - (-1)^n] \]

Simplifying,

\[ b_n = \begin{cases} 
0 & \text{n is even i.e } n = 2m \\
\frac{8LM^3}{cn \pi^3} & \text{n is odd i.e } n = 2m - 1
\end{cases} \]

The solution is the Fourier series

\[ y(x, t) = \sum_{n=1}^{\infty} \frac{8LM^3}{c(2m-1)4x^4} \sin \left( \frac{(2m-1)\pi x}{l} \right) \sin \left( \frac{(2m-1)\pi ct}{l} \right) \]

### 3.2 d’Alembert’s method of solution of the wave equation

Another method of solving the wave equation was given by d’Alembert. We will use this method to find the general solution for the wave equation, then look at the special case where the initial velocity is 0.

We have the wave equation

\[ \frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2} \]

To solve, we rewrite the wave equation in terms of 2 new independent variables -

\[ u = x + ct, \quad v = x - ct. \]

The function \( y(x, t) \) can be rewritten as a function \( y(u, v) \) of \( u \) and \( v \).

Next substitute \( y(u, v) \) into the wave equation.

Here, we want the partial derivatives of \( y(u, v) \) with respect to \( x, t \). So apply the chain rule for partial derivatives:

\[ \frac{\partial}{\partial x} y = \frac{\partial y}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial y}{\partial v} \cdot \frac{\partial v}{\partial x} \]

Substituting the partial derivatives of \( u, v \), we find that

\[ \frac{\partial}{\partial x} (y) = \frac{\partial y}{\partial u} + \frac{\partial y}{\partial v} \left( \frac{\partial u}{\partial x} + \frac{\partial v}{\partial x} \right) y \]

Then the second order derivative is

\[ \frac{\partial^2 y}{\partial x^2} = \frac{\partial}{\partial x} \left( \frac{\partial y}{\partial x} \right) = \left( \frac{\partial}{\partial u} + \frac{\partial}{\partial v} \right) \left( \frac{\partial y}{\partial x} \right) = \left( \frac{\partial}{\partial u} + \frac{\partial}{\partial v} \right) \left( \frac{\partial y}{\partial u} + \frac{\partial y}{\partial v} \right) \]
Expanding,
\[ \frac{\partial^2 y}{\partial x^2} = \frac{\partial^2 y}{\partial u^2} + 2 \frac{\partial^2 y}{\partial u \partial v} + \frac{\partial^2 y}{\partial v^2} \]

Similarly,
\[ \frac{\partial y}{\partial t} = \frac{\partial y}{\partial u} \frac{\partial u}{\partial t} + \frac{\partial y}{\partial v} \frac{\partial v}{\partial t} = c \left( \frac{\partial y}{\partial u} - \frac{\partial y}{\partial v} \right) \]
\[ \frac{\partial^2 y}{\partial t^2} = c^2 \left( \frac{\partial^2 y}{\partial u^2} - 2 \frac{\partial^2 y}{\partial u \partial v} + \frac{\partial^2 y}{\partial v^2} \right) \]

Substituting the partial derivatives in the wave equation and cancelling common terms gives the equation
\[ \frac{\partial^2 y}{\partial u \partial v} = 0 \]

We have the wave equation in terms of partial derivatives with respect to the new variables \( u \) and \( v \). This new equation, however, can be solved by direct integration.

Integrate this new equation with respect to \( v \).
\[ \int \frac{\partial^2 y}{\partial u \partial v} dv = \int 0 dv \]
\[ \frac{\partial y}{\partial u} = f(u) \]

where \( f \) is an arbitrary function of \( u \).

Integrate again, this time with respect to \( u \).
\[ \int \frac{\partial y}{\partial u} du = \int f(u) du + \psi(v) \]

where \( \psi \) is an arbitrary function of \( v \).

The integral on the right hand side is a function of \( u \), denote it by \( \phi(u) \).

So \( y = \phi(u) + \psi(v) \). Now we can substitute for \( u \) and \( v \) to get
\[ y(x, t) = \phi(x + ct) + \psi(x - ct) \]

This is the general solution of the wave equation.

Note that \( \phi \) and \( \psi \) are arbitrary functions. So we need to apply initial conditions to find them for a given problem.

We will do this for a special case, where initial displacement is given by \( f(x) \) and initial velocity is 0.
So the initial conditions are
\[ y(x, 0) = f(x) \]
\[ \frac{\partial y}{\partial t}(x, 0) = 0 \]
The solution is
\[ y(x, t) = \phi(x + ct) + \psi(x - ct) \]
\[ \frac{\partial y}{\partial t}(x, 0) = c\phi'(x + ct) - c\psi'(x - ct) \]
Applying the second initial condition, at \( t = 0 \),
\[ \frac{\partial y}{\partial t}(x, 0) = c\phi'(x) - c\psi'(x) = 0 \Rightarrow \phi'(x) = \psi'(x) \]
Integrating,
\[ \phi(x) = \psi(x) + k \]
Substitute the above in the first initial condition:
\[ y(x, 0) = \phi(x) + \psi(x) = f(x) \]
\[ y(x, 0) = \psi(x) + k + \psi(x) = 2\psi(x) + k = f(x) \]
As \( f(x) \) is a known function and \( \phi, \psi \) are arbitrary, we can use the above relation to find \( \phi \) and \( \psi \).
\[ \psi(x) = \frac{1}{2} |f(x) - k| \]
\[ \phi(x) = \psi(x) + k = \frac{1}{2} |f(x) + k| \]
Combining all of the above,
\[ y(x, t) = \phi(x + ct) + \psi(x - ct) = \frac{1}{2} [f(x + ct) + k] + \frac{1}{2} [f(x - ct) - k] \]
Simplifying further, the solution for the wave equation with initial velocity 0 and initial displacement \( f(x) \) is
\[ y(x, t) = \frac{1}{2} [f(x + ct) + f(x - ct)] \]

**Example 6.** Find the deflection of a vibrating string of unit length having fixed ends with initial velocity 0 and initial deflection \( f(x) = k(\sin x - \sin 2x) \)
Solution: By d’Alembert’s method, the solution is
\[ y(x, t) = \frac{1}{2} [f(x + ct) + f(x - ct)] \]

Substitute for \( f \):
\[ y(x, t) = \frac{k}{2} [\sin(x + ct) - \sin(2(x + ct)) + \sin(x - ct) - \sin(2(x - ct))] \]
\[ y(x, t) = \frac{k}{2} [\sin(x + ct) + \sin(x - ct) - \sin(2(x + ct)) - \sin(2(x - ct))] \]
\[ y(x, t) = \frac{k}{2} [2\sin x \cos ct - 2\sin 2x \cos 2ct] \]

The solution is
\[ y(x, t) = k [\sin x \cos ct - \sin 2x \cos 2ct] \]

4 One Dimensional Heat Flow Equation

\[ \frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \quad \text{(Heat Equation)} \]

4.1 Derivation of Heat Flow Equation

Consider the flow of heat by conduction in a uniform bar. It is assumed that the sides of the bar are insulated and the loss of heat from the sides by conduction or radiation is negligible.

Take one end of the bar as origin and the direction of the flow as the positive \( x \)-axis. The temperature \( u \) at any point of the bar depends on the distance \( x \) of the point from one end and the time \( t \). Also temperature of all points of any cross-section is the same.

The amount of heat crossing any section of the bar per second depends on the area \( A \) of the cross section, the conductivity \( K \) of the material of the bar and the temperature gradient \( \frac{\partial u}{\partial x} \) (rate of change of temperature w.r.t distance normal to the area)

\[ Q_1 = \text{The quantity of heat flowing into the section at a distance} \ x \]
\[ = -KA \left( \frac{\partial u}{\partial x} \right)_x \quad \text{per sec} \]
Figure 1:

(The negative sign on the right is attached because as \( x \) increases, \( u \) decreases)

\[
Q_2 = \text{The quantity of heat flowing into the section at a distance } z + \delta x \\
= -KA\left(\frac{\partial u}{\partial x}\right)_{x+\delta x} \text{ per sec}
\]

Hence the amount of heat retained by the slab with thickness \( \delta x \) is

\[
Q_1 - Q_2 = KA \left[ \left( \frac{\partial u}{\partial x} \right)_{x+\delta x} - \left( \frac{\partial u}{\partial x} \right)_x \right]
\]  
(1)

**But the rate of increase of heat in the slab =** \( S\rho \alpha \delta x \frac{\partial u}{\partial t} \)  
(2)

where \( S \) is the specific heat and \( \rho \), the density of the material.

From the equation (1) and (2),

\[
S\rho \alpha \delta x \frac{\partial u}{\partial t} = KA \left[ \left( \frac{\partial u}{\partial x} \right)_{x+\delta x} - \left( \frac{\partial u}{\partial x} \right)_x \right]
\]

\[
S\rho \frac{\partial u}{\partial t} = K \left[ \left( \frac{\partial u}{\partial x} \right)_{x+\delta x} - \left( \frac{\partial u}{\partial x} \right)_x \right] / \delta x
\]

Taking the limit \( \delta x \to 0 \), we have

\[
S\rho \frac{\partial u}{\partial t} = K \frac{\partial^2 u}{\partial x^2}
\]
\[
\frac{\partial u}{\partial t} = K \frac{\partial^2 u}{\partial x^2}
\]

\[
\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}
\]

where \( c^2 = \frac{K}{S\rho} \)

### 4.2 Solution of Heat Equation using separation of variables

**Objective:** Calculation of the temperature \( u \) at any point \( x \) on the bar at the time \( t \).

The **Heat equation** is \( \frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \)

Let \( u = XT \) be the solution of heat equation where \( X \) is a function of \( x \) only and \( T \) is a function of \( t \) only

i.e.

\[
\begin{align*}
    u &= XT \\
    \Rightarrow \frac{\partial u}{\partial x} &= X' T \\
    \Rightarrow \frac{\partial^2 u}{\partial x^2} &= X'' T
\end{align*}
\]

and

\[
\Rightarrow \frac{\partial u}{\partial t} = XT'
\]

Substitute these values in the heat equation \( \frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \)

\[
\Rightarrow XT' = c^2 X'' T
\]

\[
\Rightarrow \frac{X''}{X} = \frac{1}{c^2} \frac{T'}{T}
\]

The above equation can hold only when both sides reduces to a constant, say \( k \).

\[
\Rightarrow \frac{X''}{X} = \frac{1}{c^2} \frac{T'}{T} = k
\]

\[
\Rightarrow X'' = kX \text{ and } T' = kc^2 T
\]

\[
\Rightarrow X'' - kX = 0 \text{ and } T' - kc^2 T = 0
\]
Result

(i) If the roots of an auxiliary equation are complex values, say \( \alpha \pm i\beta \), then the solution is \( x = e^{\alpha x}(c_1 \cos \beta x + c_2 \sin \beta x) \)

(ii) If the roots of an auxiliary equation are equal, say \( m \), the the solution is \( x = (c_1 + c_2 x)e^{mt} \)

Case 1:- \( k \) is a positive (say \( k = p^2 \))
\[
\begin{align*}
X'' - kX &= 0 \\
X'' - p^2 X &= 0 \\
\text{Auxiliary equation is } m^2 - p^2 &= 0 \\
\Rightarrow m &= \pm p \\
\Rightarrow X &= c_1 e^{px} + c_2 e^{-px} \\
\Rightarrow T &= c_1 e^{pt} e^{ct} \\
\therefore \text{ the solution in this case is } u &= XT = (c_1 e^{px} + c_2 e^{-px})c_3 e^{p^2 t}
\end{align*}
\]

Case 2:- \( k \) is a negative (say \( k = -p^2 \))
\[
\begin{align*}
X'' - KX &= 0 \\
X'' + p^2 X &= 0 \\
\text{Auxiliary equation is } m^2 + p^2 &= 0 \\
\Rightarrow m &= \pm ip \\
\Rightarrow X &= c_1 \cos px + c_2 \sin px \\
\Rightarrow T &= c_1 e^{-p^2 t} \\
\therefore \text{ the solution in this case is } u &= XT = (c_1 \cos px + c_2 \sin px)c_3 e^{-p^2 t}
\end{align*}
\]

Case 3:- \( k = 0 \)
\[
\begin{align*}
X'' - KX &= 0 \\
X'' &= 0 \\
\text{Auxiliary equation is } m^2 &= 0 \\
\Rightarrow m &= 0, 0 \\
\Rightarrow X &= c_1 + c_2 x \\
\Rightarrow T &= c_0 \\
\therefore \text{ the solution in this case is } u &= XT = (c_1 + c_2 x)c_0
\end{align*}
\]

From the above three solution, we have to choose that solution which is consistent with the physical nature of the problem. Since \( u \) decreases as the time \( t \) increases, the
only suitable solution of the heat equation is

\[ u = XT = (c_4 \cos px + c_5 \sin px) c_6 e^{-c_7 p^2 t} \]

\[ u = (c_4 \cos px + c_5 \sin px) e^{-c_7 p^2 t} \]

\[ u = (a_x \cos px + b_x \sin px) e^{-c_7 p^2 t} \]

where \( c_4 c_5 = a_n \) and \( c_5 c_6 = b_n \)

**Example 7.** Solve the differential equation \( \frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \) for the conduction of heat along a rod without radiation, subject to the following conditions:

(i) \( u \) is not infinite for \( t \to \infty \)

(ii) \( \frac{\partial u}{\partial x} = 0 \) for \( x = 0 \) and \( x = l \)

(iii) \( u = lx - x^3 \) for \( t = 0 \), between \( x = 0 \) and \( x = l \)

Solution:- The possible solution of heat equation \( \frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \) are

(i) \( u(x, t) = (c_1 e^{px} + c_2 e^{-px}) e^{c_3 p^2 t} \)

(ii) \( u(x, t) = (c_4 \cos px + c_5 \sin px) e^{c_3 p^2 t} \)

(iii) \( u(x, t) = (c_7 + c_8 x) \)

**Condition 1** - \( u \) is not infinite for \( t \to \infty \) :- Applying this condition to the given possible solutions, i.e. when \( t \to \infty \),

(i) In solution (i) \( \Rightarrow u(x, t) = \infty \), so we reject this solution.

(ii) In solution (ii) \( \Rightarrow u(x, t) = 0 \neq \infty \)

(iii) In solution (iii) \( \Rightarrow u(x, t) \neq \infty \)

\( \therefore \) solution (ii) and (iii) satisfy condition 1

**Condition 2** - \( \frac{\partial u}{\partial x} = 0 \) for \( x = 0 \) and \( x = l \) :- Applying this condition to the acceptable solution (ii) and (iii)

solution (ii) \( \Rightarrow \frac{\partial u}{\partial x} = (-c_4 \sin px \cdot p + c_5 \cos px \cdot p) c_6 e^{-c_7 p^2 t} \)

\( \Rightarrow \frac{\partial u}{\partial x} = (-c_4 \sin px \cdot p + c_5 \cos px \cdot p) c_6 e^{-c_7 p^2 t} \)
If \( x = 0 \), then \( \frac{\partial u}{\partial x} = 0 \)
\[ \Rightarrow 0 = (c_5) p c_0 e^{-pt} \Rightarrow c_5 = 0 \]

If \( x = l \), then \( \frac{\partial u}{\partial x} = 0 \) \( \Rightarrow 0 = (-c_4 \sin px + c_5 \cos px)p c_0 e^{-pt} \)
\[ \Rightarrow 0 = (-c_4 \sin pl)p c_0 e^{-pt} \]
\[ \Rightarrow c_4 \neq 0 \text{ and } \sin pl = 0 \]
\[ \Rightarrow pl = n\pi \quad n \in \mathbb{N} \]
\[ \Rightarrow p = \frac{n\pi}{l} \]

\[ \therefore \text{ solution (ii)} \]
\[ \Rightarrow u(x,t) = \left( c_4 \cos \frac{n\pi x}{l} \right) e^{-\frac{c_4^2 n^2 \pi^2 t}{l^2}} \]
\[ \Rightarrow u(x,t) = \left( c_4 c_0 \cos \frac{n\pi x}{l} \right) e^{-\frac{c_0^2 n^2 \pi^2 t}{l^2}} \]
\[ \Rightarrow u(x,t) = \left( a_n \cos \frac{n\pi x}{l} \right) e^{-\frac{c_0^2 n^2 \pi^2 t}{l^2}} \quad \text{where } a_n = c_4 c_0 \]

The most general solution is obtained by adding all such solutions for \( n = 1, 2, 3, \ldots \)

\[ \therefore u(x,t) = \sum_{n=1}^{\infty} \left( a_n \cos \frac{n\pi x}{l} \right) e^{-\frac{c_0^2 n^2 \pi^2 t}{l^2}} \]

**Solution (iii)**: \( \frac{\partial u}{\partial x} = c_4 c_0 = 0 \) when \( x = 0 \) and \( x = l \)
\[ \Rightarrow a_0 = 0 \text{ where } c_4 c_0 = a_0 \]

Thus the general solution being the sum of solution (ii) and solution (iii)

\[ \therefore u(x,t) = a_0 + \sum_{n=1}^{\infty} \left( a_n \cos \frac{n\pi x}{l} \right) e^{-\frac{c_0^2 n^2 \pi^2 t}{l^2}} \]

**Condition 3**: \( u = lx - x^2 \) for \( t = 0 \), between \( x = 0 \) and \( x = l \) i.e. \( u(x,0) = lx - x^2 \)

put \( t = 0 \) in the above equation \( \Rightarrow u(x,0) = a_0 + \sum_{n=1}^{\infty} \left( a_n \cos \frac{n\pi x}{l} \right) \)
\[ \Rightarrow lx - x^2 = a_0 + \sum_{n=1}^{\infty} \left( a_n \cos \frac{n\pi x}{l} \right) \] which is the expansion of \( lx - x^2 \) as a half range cosine series in \( (0,l) \)
\[ a_0 = \frac{2}{l} \int_{0}^{l} (lx - x^2) \, dx = \frac{2}{l} \left( \frac{lx^2}{2} - \frac{x^3}{3} \right)_{0}^{l} = \frac{2}{l} \left[ \frac{l^3}{2} - \frac{l^3}{3} \right] = \frac{l^2}{3} \]
\[ a_n = \frac{2}{l} \int_0^l \left( l - x^2 \right) \cos \frac{n \pi x}{l} \, dx = \frac{2}{l} \left[ \left( l - x^2 \right) \sin \frac{n \pi x}{l} \frac{l}{n \pi} \left( \left( -\cos \frac{n \pi x}{l} \frac{l^2}{n^2 \pi^2} \right) + \left( -2 \left( -\sin \frac{n \pi x}{l} \frac{l^3}{n^3 \pi^3} \right) \right) \right]_0^l = \frac{2}{l} \left[ \frac{l^3}{n^2 \pi^2} \frac{\cos n \pi - 1}{n \pi} \right] = -\frac{2l^2}{n^2 \pi^2} \left( -1 \right)^n + 1 \]

\[ \therefore u(x, t) = \frac{l^2}{6} + \sum_{n=1}^{\infty} \left( \frac{-2l^2}{n^2 \pi^2} \left( -1 \right)^n + 1 \right) \cos \frac{n \pi x}{l} \frac{\cos \frac{n \pi t}{l}}{l^2} \]

**Case 1:** Both the ends are kept at 0°C temperature

**Example 8.** A rod of length \( l \) with insulated sides is initially at a uniform temperature \( u_0 \). Its ends are suddenly cooled to 0°C and are kept at that temperature. Find the temperature function \( u(x, t) \).

**Solution:** The temperature function \( u(x, t) \) satisfies the differential equation

\[ \frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \]

and we know

\[ u(x, t) = (a_n \cos px + b_n \sin px)e^{-\frac{c^2 \pi^2}{l^2} t} \]  \hspace{1cm} \text{(3)}

\[ u = 0 \hspace{1cm} u_0 \hspace{1cm} x = l \hspace{1cm} u = 0 \]

Applying the boundary conditions in equation (3)

**Case 1:** \( \text{At} \ x = 0, \ u = 0 \)

\[ \Rightarrow u(0, t) = (a_n \cos 0 + b_n \sin 0)e^{-\frac{c^2 \pi^2}{l^2} t} \]

\[ \Rightarrow 0 = a_n e^{-\frac{c^2 \pi^2}{l^2} t} \]

\[ \Rightarrow a_n = 0 \]

\[ \therefore \text{equation (3)} \]

\[ \Rightarrow u(x, t) = (b_n \sin px)e^{-\frac{c^2 \pi^2}{l^2} t} \]  \hspace{1cm} \text{(4)}

**Case 2:** \( \text{At} \ x = l, \ u = 0 \)

\[ \Rightarrow u(l, t) = (b_n \sin pl)e^{-\frac{c^2 \pi^2}{l^2} t} \]

\[ \Rightarrow 0 = (b_n \sin pl)e^{-\frac{c^2 \pi^2}{l^2} t} \]
\[ b_n \neq 0 \text{ and } \sin \frac{n\pi}{l} = 0 \quad \text{[if } b_n = 0 \text{ then } u(x, t) = 0] \]
\[ pl = n\pi \text{ where } n \in N \]
\[ p = \frac{n\pi}{l} \text{ where } n \in N \]
\[ \therefore \text{ equation (4)} \]

\[ u(x, t) = (b_n \sin \frac{n\pi}{l} x) e^{-\frac{c^2 n^2 \pi^2 t}{l^2}} \quad (5) \]

The most general solution is obtained by adding all such solutions for \( n = 1, 2, 3, ... \)
\[ \therefore \text{ equation (5)} \]

\[ u(x, t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{l} x \cdot e^{-\frac{c^2 n^2 \pi^2 t}{l^2}} \quad (6) \]

**Case 3** Applying the initial condition \( u(x, 0) = u_0 \text{ at } t = 0, u = u_0 \) in equation (6)

\[ \Rightarrow u(x, 0) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{l} x \cdot e^{0} \]
\[ \Rightarrow u_0 = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{l} x \]

which is the half range sine series for the function \( u_0 \).

**Result**

The half range sine series expansion for the function \( f(x) \) is represented in the interval \((0, l)\) as \( f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \) where \( b_n = \frac{2}{l} \int_{0}^{l} u_0 \sin \frac{n\pi x}{l} \, dx \)

\[ \Rightarrow b_n = \frac{2}{l} \int_{0}^{l} u_0 \sin \frac{n\pi x}{l} \, dx = \frac{2}{l} u_0 \left( \cos \frac{n\pi x}{l} \right) \bigg|_{0}^{l} \]
\[ \Rightarrow b_n = \frac{2}{l} u_0 \left( \cos \frac{n\pi l}{l} - \cos \frac{n\pi 0}{l} \right) \]
\[ \Rightarrow b_n = \frac{2u_0}{n\pi} \left( \cos n\pi - \cos 0 \right) \quad \left[ \cos n\pi = (-1)^n \right] \]
\[ \Rightarrow b_n = \frac{2u_0}{n\pi} \left( (-1)^n - 1 \right) \]

Hence the temperature function, from the equation (6)

\[ \Rightarrow u(x, t) = \sum_{n=1}^{\infty} \frac{2u_0}{n\pi} \left( (-1)^n - 1 \right) \sin \frac{n\pi x}{l} \cdot e^{-\frac{c^2 n^2 \pi^2 t}{l^2}} \]
\[ u(x, t) = \frac{2u_0}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} ((-1)^n - 1) \sin \left( \frac{n\pi}{l} x \right) e^{-\frac{cn^2\pi^2 t}{l^2}} \]

**Example 9.** A uniform bar of length \( l \) through which heat flows is insulated at its sides. The ends are kept at zero temperature. The initial temperature at the interior points are given by \( f(x) = k \sin^3 \left( \frac{\pi x}{l} \right) \). Find the temperature distribution in the bar after time \( t \).

**Solution:** The temperature function \( u(x, t) \) satisfies the differential equation

\[ \frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2} \]

and we know

\[ u(x, t) = (a_x \cos px + b_x \sin px) e^{-c^2pt} \]

From example 8

\[ u(x, t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{l} x \cdot e^{-\frac{c^2n^2\pi^2 t}{l^2}} \]

Applying the initial condition \( u(x, 0) = k \sin^3 \left( \frac{\pi x}{l} \right) \) \( \text{at} \ t = 0, \ u = k \sin^3 \left( \frac{\pi x}{l} \right) \)

\[ u(x, 0) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{l} x \cdot e^0 \]

\[ k \sin^3 \left( \frac{\pi x}{l} \right) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi}{l} x \]

\[ \Rightarrow \frac{k}{4} \left( 3 \sin \frac{\pi x}{l} - \sin \frac{3\pi x}{l} \right) = b_1 \sin \frac{\pi x}{l} + b_2 \sin \frac{2\pi x}{l} + b_3 \sin \frac{3\pi x}{l} + b_4 \sin \frac{4\pi x}{l} + \ldots \]

\[ \Rightarrow b_1 = \frac{3k}{4}, \quad b_2 = 0, \quad b_3 = -\frac{k}{4}, \quad b_n = 0 \forall n > 3 \]

\[ u(x, t) = b_1 \sin \frac{\pi x}{l} \cdot e^{-\frac{c^2\pi^2 t}{l^2}} + b_2 \sin \frac{2\pi x}{l} \cdot e^{-\frac{4c^2\pi^2 t}{l^2}} + b_3 \sin \frac{3\pi x}{l} \cdot e^{-\frac{9c^2\pi^2 t}{l^2}} \]

\[ u(x, t) = \frac{3k}{4} \sin \frac{\pi x}{l} \cdot e^{-\frac{c^2\pi^2 t}{l^2}} - \frac{k}{4} \sin \frac{3\pi x}{l} \cdot e^{-\frac{9c^2\pi^2 t}{l^2}} \]
Result

(i) \( \sin^3 x = \frac{3 \sin x - \sin 3x}{4} \)

(ii) \( \sin A \sin B = \frac{1}{2} (\cos (A - B) - \cos (A + B)) \)

References

MAT 201

RAJAGIRI SCHOOL OF ENGINEERING & TECHNOLOGY
DEPARTMENT OF MATHEMATICS
TUTORIAL / ASSIGNMENT/ UNIT-WISE QUESTION BANK RECORD BOOK
COURSE:- MAT 201; PARTIAL DIFFERENTIAL EQUATIONS AND COMPLEX ANALYSIS
Branch: COMMON FOR ALL BRANCHES (EXCEPT CS & IT)

MODULE III COMPLEX VARIABLE - DIFFERENTIATION

TUTORIAL

1. Show that \( f(z) = \begin{cases} \frac{(x + y)^2}{x^2 + y^2} & z \neq 0 \\ 0 & z = 0 \end{cases} \) is discontinuous at \( z = 0 \).

2. Show that \( f(z) = \begin{cases} \frac{xy^2(x + iy)}{x^2 + y^4} & z \neq 0 \\ 0 & z = 0 \end{cases} \) is not differentiable at \( z = 0 \).

3. Find an analytic function whose real part is \( e^z \cos y \).

4. Find the points at which the function \( w = \cos z \) is not conformal

5. Find the image of the circle \( x^2 + y^2 + ay = 0 \) under the transformation \( w = \frac{1}{z} \)

ASSIGNMENT

1. Show that \( \lim_{z \to 0} \frac{x^2 y}{z^2 + y^2} \) does not exist though this function approaches the same limit along every straight line through the origin.

2. Find out, and give reason, whether

\[
f(z) = \begin{cases} \frac{(Re z^2)/|z|}{z \neq 0} \\ 0 & z = 0 \end{cases}
\]

is continuous at \( z = 0 \).

3. Show that \( w = \sin z \) is analytic everywhere. Also find its derivative.

4. Show that

\[
f(z) = \begin{cases} \frac{x^3 (1 + i) - y^3 (1 - i)}{x^2 + y^2} & z \neq 0 \\ 0 & z = 0 \end{cases}
\]

satisfies the Cauchy-Riemann equations at \( z = 0 \), but not differentiable at \( z = 0 \).

5. Find the value of \( a \) so that \( u = xy + ax^3 - y^3 \) is harmonic. Find its harmonic conjugate.

6. Find the harmonic conjugate of \( u = \frac{x}{x^2 + y^2} \)

7. Find the critical point and fixed point of \( w = \frac{1}{2} \left( z + \frac{1}{z} \right) \)

8. Discuss the transformation \( w = \cos z \)
9. Find all linear fractional transformation with fixed points $z = 0$

10. (a) Give an example of a harmonic function which is not analytic
    (b) Give an example linear fractional transformation which have no fixed points.

**UNIT-WISE QUESTION BANK**

1. Show that the function
   
   \[ f(z) = \begin{cases} \frac{2xy^2}{x^2 + 3y^2} & z \neq 0 \\ 0 & z = 0 \end{cases} \]
   
   is discontinuous at $z = 0$.

2. Show that the function $f(z) = \sinh z$ is analytic everywhere. Also find its derivative.

3. Show that the function $f(z) = \sqrt{x+y}$ is not differentiable at $z = 0$.

4. Prove that the function
   
   \[ f(z) = \begin{cases} \frac{x^2}{y^2} (x + iy) & z \neq 0 \\ 0 & z = 0 \end{cases} \]
   
   is not differentiable at origin, although Cauchy-Riemann equations are satisfied at the origin.

5. Prove that $u = x^2 + y^2$ is not a harmonic function.

6. Find an analytic function whose imaginary part is $\frac{x}{x^2 + y^2}$.

7. Find $a$ so that the function $u = x^3 + axy^2$ is harmonic. Find its harmonic conjugate.

8. If $u$ is harmonic prove that $u_x - iu_y$ is analytic.

9. Find all points at which the function $w = \cosh 2\pi z$ is not conformal

10. Find all linear fractional transformation with fixed points $z = \pm i$

11. Find all linear fractional transformation with fixed points $z = \pm 1$

12. Find the fixed points of the transformation $w = \frac{iz + 4}{2z - 5i}$

13. Find the fixed points of the transformation $w = \frac{az - 1}{z + ai}$

14. Find the inverse of the inverse of the transformation $w = \frac{i}{2z - 1}$

15. Find the inverse of the inverse of the transformation $w = \frac{z - i}{-iz/2 - 1}$
Pre requisites

(i) Complex arithmetic - addition, subtraction, multiplication, conjugate and division

(ii) Complex plane and Argand diagram

(iii) Polar representation of complex numbers

(iv) Euler’s formula, De Moivre's formula

(v) Distance formula in the complex plane

(vi) Exponential representation of trigonometric and hyperbolic functions
In this module, we will look at complex functions, differentiation of complex functions and some special conformal maps.

1 Sets and curves in the complex plane

We will require knowledge of some special sets in the complex plane for both Modules 4 and 5.

They are:

(i) **General circle of radius $\rho$ and center $a$.**

The set of all points of this circle is given by $\{z : |z - a| = \rho\}$.

An important special case is when $\rho = 1, a = 0$. In this case we have a circle given by $|z| = 1$, called the **unit circle**

(ii) **Disc of radius $\rho$ and center $a$.**

The set given by $\{z : |z - a| < \rho\}$ is called the **open circular disc**. This set contains all the points enclosed by the circle $\{z : |z - a| = \rho\}$, but excluding the circle itself.

The set given by $\{z : |z - a| \leq \rho\}$ is called the **closed circular disc**. This set contains all the points enclosed by the circle $\{z : |z - a| = \rho\}$ as well as the circle itself.

(iii) **Annulus with center $a$, inner radius $\rho_1$ and outer radius $\rho_2$.**

The open annulus, given by $\{z : \rho_1 < |z - a| < \rho_2\}$ is the set of all complex numbers $z$ whose distance from $a$ is greater than $\rho_1$ and less than $\rho_2$

(iv) **Half Planes**

As the name suggests, half planes cover half of the entire complex plane / Argand plane. There are 4 possible half planes:
(a) Upper half plane: \( \{ z = x + iy \in \mathbb{C} | y > 0 \} \)
(b) Lower half plane: \( \{ z = x + iy \in \mathbb{C} | y < 0 \} \)
(c) Right half plane: \( \{ z = x + iy \in \mathbb{C} | x > 0 \} \)
(d) Left half plane: \( \{ z = x + iy \in \mathbb{C} | x < 0 \} \)

2 Complex functions

Definition 2.1. Let \( S \) be a set of complex numbers. A complex function on \( S \) is a rule that assigns to every \( z \) in \( S \) a complex number \( w \), called the value of \( f \) at \( z \).

We write this as

\[ w = f(z) \]

\( z \) varies in \( S \) and is called a complex variable. \( S \) is called the domain of \( f \).

Example 1. \( f(z) = z^2 + 3z \) is a complex function for all \( z \), that is, its domain is the whole complex plane.

Example 2. \( w = f(z) = \frac{1}{z} \) is not defined for \( z = 0 \), so its domain is \( \mathbb{C}/\{0\} \)

Definition 2.2. The set of all values \( w \) of a function \( f \) is called the range of \( f \).

\( w \) is a complex number, which we write as \( w = f(z) = u + iv \), where \( u, v \) are real numbers. But \( w \) depends on \( z = x + iy \). So \( u, v \) become real valued 2 variable functions of \( x \) and \( y \).

Result:

Every complex function can be written as \( w = f(z) = u(x, y) + iv(x, y) \)

The function \( u(x, y) \) is called the real part of the function \( f \), and \( v(x, y) \) the imaginary part of the function \( f \).
Example 3. Let $f(z) = z^2 + 3z$. Find $u$ and $v$. Calculate the value of $f$ at $z = 1 + 3i$.

Solution: We can write the given function as

$$f(z) = (x + iy)^2 + 3(x + iy).$$

Expanding terms,

$$f(z) = x^2 - y^2 + 2xy + 3x + 3y = (x^2 - y^2 + 3x) + i(2xy + 3y)$$

Therefore

$$u(x, y) = x^2 - y^2 + 3x, v(x, y) = 2xy + 3y$$

Finally, $f(1 + 3i) = u(1, 3) + iv(1, 3) = -5 + 15i$

3 Limit, continuity and differentiation

Definition 3.1. A function $f(z)$ is said to have limit $l$ as $z$ approaches $z_0$, if $f$ is defined in a neighborhood of $z_0$, and for every real $\epsilon > 0$, we can find $\delta > 0$ such that

$$|f(z) - l| < \epsilon \text{ in } |z - z_0| < \delta$$

This is written as

$$\lim_{z \to z_0} f(z) = l$$

In the real case, a point $x$ could approach $x_0$ only from 2 directions - positive or negative.

But in the complex case, $z$ could approach $z_0$ along infinitely many directions or curves. For the limit to exist, the value $l$ has to remain the same in every direction, that is, limit should be independent of path.

Definition 3.2. A function $f(z)$ is said to be continuous at $z = z_0$ if $f(z_0)$ is defined, and $\lim_{z \to z_0} f(z) = f(z_0)$. 
A function $f$ is continuous in a domain if it is continuous at each point in the domain.

**Example 4.** Is the function $f(z) = \begin{cases} \frac{z}{z} & z \neq 0 \\ 0 & z = 0 \end{cases}$ continuous at $z = 0$?

Solution: For $f(z)$ to be continuous at 0, we require $\lim_{z \to 0} f(z) = f(0) = 0$.

Let us evaluate this limit:

$$\lim_{z \to 0} f(z) = \lim_{z \to 0} \frac{x + iy}{z - iy}$$

Remember that the limit should be the same as we approach 0 from any direction. Let $z \to 0$ along the straight line $y = mx$ with slope $m$ ($m$ is a real number).

Then the limit becomes

$$\lim_{x \to 0} \frac{x + imx}{x - imx} = \lim_{x \to 0} \frac{1 + im}{1 - im} = \frac{1 + im}{1 - im}$$

The limit as $z \to 0$ along the line $y = mx$ is the complex number $\frac{1 + im}{1 - im}$.

As $m$ changes, we are approaching 0 from different directions. However, the limit we found depends on the slope $m$.

As this value is different for different directions, the limit does not exist.

Since the limit does not even exist, this function is not continuous at 0.

**Definition 3.3.** The **derivative** of a complex function $f$ at a point $z_0$ is written $f'(z_0)$, and is defined by

$$f'(z_0) = \lim_{z \to z_0} \frac{f(z) - f(z_0)}{z - z_0} = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z}$$

Notice that the derivative is also given by a limit, so for a function to be differentiable, this limit should exist.

**Example 5.** $f(z) = z^2$ is differentiable for all $z$ and has derivative $f'(z) = 2z$.

Solution:

By the definition of the derivative

$$f'(z) = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z}$$
\[ f'(z) = \lim_{\Delta z \to 0} \frac{(z + \Delta z)^2 - z^2}{\Delta z} \]

Expanding the numerator and simplifying,

\[ f'(z) = \lim_{\Delta z \to 0} \frac{z^2 + 2z\Delta z + \Delta z^2 - z^2}{\Delta z} = \lim_{\Delta z \to 0} \frac{2z + \Delta z}{\Delta z} = \lim_{\Delta z \to 0} 2z + \Delta z \]

Applying the limit,

\[ f'(z) = 2z \]

**Example 6.** Show that \( f(z) = \bar{z} \) is not differentiable.

**Solution:**

To apply the definition of the derivative, we need to look at the term \( \frac{f(z + \Delta z) - f(z)}{\Delta z} \).

For the given function, this is

\[ \frac{f(z + \Delta z) - f(z)}{\Delta z} = \frac{z + \Delta z - \bar{z}}{\Delta z} \]

Simplifying,

\[ \frac{f(z + \Delta z) - f(z)}{\Delta z} = \frac{\Delta z}{\Delta z} = \frac{\Delta x - i\Delta y}{\Delta x + i\Delta y} \]

To show that the limit does not exist, we show that we get different values for the limit when we let \( \Delta z \) go to 0 along different paths.

Consider the following 2 paths:

![Diagram](image)

If we let \( \Delta z \to 0 \) along path 1, first \( \Delta y \to 0 \), and then \( \Delta x \to 0 \).

Applying these limits

\[ \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z} = \lim_{\Delta x \to 0} \lim_{\Delta y \to 0} \frac{\Delta x - i\Delta y}{\Delta x + i\Delta y} \]
After applying the first limit
\[
\lim_{\Delta x \to 0} \frac{f(z + \Delta x) - f(z)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta x}{\Delta x}
\]
This is just
\[
\lim_{\Delta x \to 0} 1 = 1
\]
Along path II, first \( \Delta x \to 0 \), and then \( \Delta y \to 0 \).
If we let \( \Delta x \to 0 \) along this path, the limit is \(-1\).
Along 2 different paths, we get 2 different limits. So the limit does not exist.
Therefore the derivative does not exist, and \( f(z) = z \) is not differentiable.

Alternate method : We could let \( \Delta x \) approach 0 along \( y = nx \) and show that
the limit does not exist, as in Example 4

## 3.1 Rules of differentiation

The rules for differentiation for complex differentiation are the same as for real functions.

Let \( c \) be any (complex) constant, and \( f, g \) differentiable functions. Then

(i) \( (cf)' = cf' \)

(ii) \( (f + g)' = f' + g' \)

(iii) \( (fg)' = f'g + fg' \)

(iv) \( \left( \frac{f}{g} \right)' = \frac{f'g - fg'}{g^2} \)

(v) Power rule : \( (z^n)' = nz^{n-1} \) (we proved this for \( n = 2 \))

(vi) The chain rule differentiation of composition of complex functions is the same as
that for differentiation of real valued functions.

(vii) As in the real case, if \( f \) is differentiable at a point, it is also continuous there.
4 Analytic functions

Definition 4.1. A function $f(z)$ is said to be analytic in a domain $D$ if $f(z)$ is defined and differentiable at all points of $D$. $f(z)$ is said to be analytic at a point $z = z_0$ in $D$ if $f(z)$ is analytic in a small open disc containing $z_0$.

Another term for analytic functions is holomorphic function.

Example 7. Non-negative integer powers of $z$: $1, z, z^2, \ldots$ are analytic in the entire complex plane.

Example 8. All polynomial functions: Functions of the form $f(z) = c_0 + c_1 z + c_2 z^2 + \cdots + c_n z^n$, where $c_0, c_1, \ldots, c_n$ are complex constants, are analytic.

Example 9. Rational functions: If $h(z), g(z)$ are polynomials with no common factors, $f(z) = \frac{g(z)}{h(z)}$ is called a rational function.

This function is analytic at all points except those where $h(z) = 0$.

5 Cauchy - Riemann equations

The Cauchy - Riemann equations are the most important equations in this module.

They provide a test for analyticity of a complex function.

Theorem 5.1. Let $f(z) = u(x, y) + iv(x, y)$ be defined and continuous in some neighborhood of a point $z$ and differentiable at $z$ itself.

Then $u$ and $v$ satisfy the Cauchy - Riemann equations

$$u_x = v_y, \quad u_y = -v_x$$

at that point.

Remark 1. Hence, if $f(z)$ is analytic on a domain $D$, these partial derivatives exist and satisfy the Cauchy - Riemann equations at all points of $D$.

Example 10. Derive the necessary conditions for a complex function $f(z)$ to be analytic at a point $z$. 

Solution: The necessary conditions for \( f(z) = u(x, y) + iv(x, y) \) to be analytic is the Cauchy-Riemann equations:

\[
\begin{align*}
    u_x &= v_y, \quad u_y = -v_x
\end{align*}
\]

Since \( f(z) \) is differentiable, \( f'(z) \) exists, and is given by

\[
    f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z}
\]

where \( \Delta z = \Delta x + i\Delta y \)

So

\[
    z + \Delta z = x + \Delta x + i(y + \Delta y)
\]

\[
    f(z + \Delta z) - f(z) = u(x + \Delta x, y + \Delta y) + iv(x + \Delta x, y + \Delta y) - u(x, y) - iv(x, y)
\]

The idea of the proof is similar to what we did in Example 6: Let \( \Delta z \to 0 \) along different paths. Since \( f'(z) \) exists, the limit should be same along all these different paths.

\[
    \frac{f(z + \Delta z) - f(z)}{\Delta z} = \frac{u(x + \Delta x, y + \Delta y) - u(x, y) + iv(x + \Delta x, y + \Delta y) - iv(x, y)}{\Delta x + i\Delta y}
\]  \( \text{(1)} \)

(i) Let \( \Delta z \to 0 \) along path I.

So first \( \Delta y \to 0 \), and then \( \Delta x \to 0 \).

\[
    f'(z) = \lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z} = \lim_{\Delta x \to 0} \lim_{\Delta y \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z}
\]

Now using equation 1 and applying the first limit, this becomes
\[ f'(z) = \lim_{\Delta x \to 0} \frac{u(x + \Delta x, y) - u(x, y)}{\Delta x} + i \frac{v(x + \Delta x, y) - v(x, y)}{\Delta x} \]

By the definition of partial derivatives, this is just

\[ f'(z) = u_x + iv_x \quad (2) \]

(ii) Let \( \Delta x \to 0 \) along path II.

Here, first \( \Delta x \to 0 \), and then \( \Delta y \to 0 \).

Repeating the same steps as before,

\[ f'(z) = \lim_{\Delta y \to 0} \frac{u(x, y + \Delta y) - u(x, y)}{i\Delta y} + i \frac{v(x, y + \Delta y) - v(x, y)}{i\Delta y} \]

Again by the definition of partial derivatives,

\[ f'(z) = \frac{u_y}{i} + v_y \]

Since \( \frac{i}{i} = -i \), this becomes

\[ f'(z) = v_y - iu_y \quad (3) \]

Since \( f(z) \) is differentiable, \( f'(z) \) exists.

Therefore the expressions for \( f'(z) \) given by 2 and 3 should be equal.

\[ u_x + iv_x = u_y - iu_y \]

Equating the real and imaginary parts of the above complex numbers gives the necessary conditions for a function \( f \) to be analytic.

The conditions are

\[ u_x = u_y, \quad u_y = -v_x \]

These are just the Cauchy - Riemann equations.
Example 11. \( f(z) = z^2 \) is analytic for all \( z \). So, it should satisfy the Cauchy-Riemann equations.

We can verify this: For this function \( u = x^3 - y^3, v = 2xy \).

\[
\begin{align*}
  u_x &= 2x, & u_y &= -2y \\
  v_x &= 2y, & u_y &= 2x
\end{align*}
\]

These partial derivatives satisfy the Cauchy-Riemann equations.

Example 12. For \( f(z) = \bar{z}, u = x, v = -y \).

This \( u \) and \( v \) don’t satisfy the Cauchy-Riemann equations anywhere. Therefore, it is not analytic.

5.1 The Cauchy-Riemann equations are necessary, but not sufficient for analyticity

Example 13. Show that the function \( f(z) = \sqrt{|xy|} \) is not differentiable at the origin, even though it satisfies the Cauchy-Riemann equations at the origin.

Solution:

For this function \( u = \sqrt{|xy|}, \quad v = 0 \).

Evaluate the partial derivatives of \( u \) and \( v \).

Since \( v = 0 \) (constant function), \( v_x = 0, \quad v_y = 0 \).

To evaluate \( u_x \) and \( u_y \) at the origin, we use the definition of the partial derivatives:

\[
\begin{align*}
  u_x(0, 0) &= \lim_{x \to 0} \frac{u(x, 0) - u(0, 0)}{x - 0} = \lim_{x \to 0} \frac{0 - 0}{x} = 0 \\
  u_y(0, 0) &= \lim_{y \to 0} \frac{u(0, y) - u(0, 0)}{y - 0} = \lim_{y \to 0} \frac{0 - 0}{y} = 0
\end{align*}
\]

Looking at the partial derivatives, it is clear that the Cauchy-Riemann equations are satisfied at the origin.

However, this function is not differentiable at the origin.

If \( f'(0) \) existed, it would be given by

\[
\lim_{z \to 0} \frac{f(z) - f(0)}{z}
\]
Substituting for $f$, this limit is

$$\lim_{z \to 0} \sqrt{|xy|}$$

Let $z \to 0$ along the straight line $y = nx$.

Then this limit is

$$\lim_{z \to 0} \sqrt{|mx^2|} = \lim_{z \to 0} \sqrt{|m|}$$

As the limit depends on slope $m$, it does not exist.

So this function is not differentiable at the origin, even though it satisfies the Cauchy-Riemann equations at this point.

6 Laplace’s equation and Harmonic functions

The importance of complex analysis in engineering comes from the fact that both the real and imaginary parts of an analytic function are solutions of the Laplace’s equation, a second order PDE found in gravitation, electrostatics, fluid flow and heat conduction.

Laplace’s equation

$$\nabla^2 z = z_{xx} + z_{yy} = 0$$

Theorem 6.1. If $f(z) = u(x, y) + iv(x, y)$ is analytic on a domain $D$, then both $u$ and $v$ satisfy the Laplace’s equation.

$$\nabla^2 u = u_{xx} + u_{yy} = 0$$

$$\nabla^2 v = v_{xx} + v_{yy} = 0$$

Proof:

Since $f$ is analytic, $u$ and $v$ should satisfy the Cauchy-Riemann equations:

$$u_x = v_y, \quad u_y = -v_x$$

If we evaluate the second order derivatives of $u$, we find

$$u_{xx} = u_{xy}, \quad u_{yy} = -v_{xx} = -v_{xy}$$
Then
\[ u_{xx} + u_{yy} = v_{xx} - v_{yy} = 0 \]

Therefore, the real part \( u \) of \( f \) satisfies the Laplace's equation.

A similar calculation will show that \( v \) also satisfies the Laplace's equation.

**Definition 6.1.** Functions which satisfy Laplace’s equation are called **harmonic functions**.

**Example 14.** The real and imaginary parts of an analytic function are harmonic.

**Definition 6.2.** If \( u \) and \( v \) are harmonic functions such that \( f = u + iv \) is analytic, then \( v \) is called the **harmonic conjugate** of \( u \).

### 6.1 Finding the harmonic conjugate

Given a function \( u \), we are interested in finding its harmonic conjugate. The steps involved in this calculation are:

(i) Check that \( u \) is harmonic, that is, it satisfies the Laplace’s equation.

(ii) Let the harmonic conjugate of \( u \) be \( v \). Then \( f = u + iv \) is analytic.

(iii) \( u \) and \( v \) should satisfy the Cauchy-Riemann equations \( u_x = v_y, u_y = -v_x \)

(iv) Integrate \( v_y \) with respect to \( x \) (or \( v_x \) with respect to \( y \)) to find \( v(x, y) \). This \( v \) will include an arbitrary function of \( y \) (arbitrary function of \( x \), if you had integrated with respect to \( y \)).

(v) Now use the remaining Cauchy-Riemann equation to find the arbitrary function

**Example 15.** Find the harmonic conjugate of \( u = x^2 - y^2 - y \)

Solution: For the given function,

\[
  u_x = 2x, \quad u_y = -2y - 1
\]

\[
  u_{xx} = 2, u_{yy} = -2
\]
\[ u_{xx} + u_{yy} = 2 - 2 = 0 \]

So, \( u \) is a harmonic function.

Let \( v \) be the harmonic conjugate of \( u \).

Then \( u + iv \) is analytic, and \( u, v \) satisfy the Cauchy-Riemann equations

\[ u_x = v_y, \quad u_y = -v_x \]

If we substitute for the partial derivatives of \( u \),

\[ v_x = 2y + 1, \quad v_y = 2x \]

Then

\[ v(x, y) = \int v_x \, dx \]

\[ v(x, y) = \int (2y + 1) \, dx = (2y + 1)x + \phi(y) \quad (4) \]

where \( \phi \) is an arbitrary function of \( y \).

Use the remaining condition \( v_y = 2x \):

Differentiate equation 4 with respect to \( y \) and apply the above condition.

\[ v_y = 2x + \phi'(y) = 2x \]

From this

\[ \phi'(y) = 0 \]

Therefore

\[ \phi(y) = c \]

where \( c \) is an arbitrary constant.

To conclude, the harmonic conjugate of \( u = x^2 - y^2 - y \) is \( v = (2y + 1)x + c \).

**Example 16.** Find an analytic function whose real part is \( e^x \cos y \)
Example 17. Discuss the conformality of $f(z) = z^3$

$f(z) = z^3$ has critical point at $z = 0$ since $f(z) = 2z = 0 \Rightarrow z = 0$ and angles are doubled, so conformality fails.

Example 18. Discuss the conformal mapping $w = z^2$

$w = z^2$ is conformal at all points in the complex plane except $z = 0$. (From example 17)

$$
\begin{align*}
    w &= z^2 \\
    u + iv &= (x + iy)^2 = x^2 - y^2 + i2xy \\
    \Rightarrow u &= x^2 - y^2 \text{ and } v = 2xy
\end{align*}
$$

Case 1: Image of real axis $y = 0$ in the $z$ plane under $w = z^2$

$y = 0 \Rightarrow u = x^2$ and $v = 0$ which is the positive real axis in the $w$ plane. That is under the transformation $w = z^2$, the points on the real axis of $z$ plane are mapped into positive real axis of $w$-plane.

Case 2: Image of imaginary axis $x = 0$ in the $z$ plane under $w = z^2$

$x = 0 \Rightarrow u = -y^2$ and $v = 0$ which is the negative real axis in the $w$ plane. That
is under the transformation $w = z^2$, the points on the imaginary axis of $z$ plane are mapped into negative real axis of $w$-plane.

**Case 3:** Image of lines $x = c$, parallel to the imaginary axis of $z$ plane under $w = z^2$

Now $x = c \Rightarrow u = c^2 - y^2$ and $v = 2cy \Rightarrow y = \frac{v}{2c}$.

Eliminating $y$, $u = c^2 - \frac{v^2}{4c^2} \Rightarrow v^2 = -4c^2(u - c^2)$ which is a parabola whose vertex at $(c^2, 0)$

**Case 4:** Image of lines $y = c$, parallel to the real axis of $z$ plane under $w = z^2$

Now $y = c \Rightarrow u = x^2 - c^2$ and $v = 2cx \Rightarrow x = \frac{v}{2c}$.

Eliminating $x$, $u = c^2 - \frac{v^2}{4c^2} \Rightarrow v^2 = 4c^2(u + c^2)$ which is a parabola whose vertex at $(-c^2, 0)$

This show that the line parallel to he real and imaginary axis are transformed into parabolas in $w$-plane.

**Example 19. Discuss the transformation $w = e^z$**

Let $w = e^z \Rightarrow u + iv = e^{x+iy} = e^x e^{iy} = e^x (\cos y + i \sin y)$

$\Rightarrow u = e^x \cos y \text{ and } v = e^x \sin y$

**Case 1:** Image of real axis $x = 0$ in the $z$ plane under $w = e^z$

$y = 0 \Rightarrow u = e^x \text{ and } v = 0$ which is the positive real axis in the $w$ plane. That is under the transformation $w = e^z$, the points on the real axis of $z$ plane are mapped into positive real axis of $w$-plane.

**Case 2:** Image of imaginary axis $x = 0$ in the $z$ plane under $w = e^z$

$x = 0 \Rightarrow u = \cos y \text{ and } v = \sin y \Rightarrow u^2 + v^2 = 1$ which is the unit circle in the $w$-plane. That is under the transformation $w = e^z$, the points on the imaginary axis of $z$ plane are mapped into unit circle of $w$-plane.

**Case 3:** Image of lines $x = c$, parallel to the imaginary axis of $z$ plane under $w = e^z$

Now $x = c \Rightarrow u = e^c \cos y \text{ and } v = e^c \sin y \Rightarrow \cos y = \frac{u}{e^c} \text{ and } \sin y = \frac{v}{e^c}$.

Then $\cos^2 y + \sin^2 y = \frac{u^2}{(e^c)^2} + \frac{v^2}{(e^c)^2} = 1 \Rightarrow u^2 + v^2 = (e^c)^2$ which is a circle with center at origin and radius $e^c$.

**Case 4:** Image of lines $y = c$, parallel to the real axis of $z$ plane under $w = e^z$

Now $y = c \Rightarrow u = e^c \cos c \text{ and } v = e^c \sin c \Rightarrow \cos c = \frac{u}{e^c} \text{ and } \sin c = \frac{v}{e^c}$.

$\Rightarrow \tan c = \frac{v}{u} \Rightarrow v = u \tan c$ which is straight line passing through origin.

**Example 20. Discuss the transformation $w = \sin z$**
Let \( w = \sin z = \sin(x + iy) = \sin x \cosh y + i \cos x \sinh y \)
\( u = \sin x \cosh y \) and \( v = \cos x \sinh y \)

**Case 1:** Image of real axis \( y = 0 \) in the \( z \) plane under \( w = \sin z \)
\( y = 0 \Rightarrow u = \sin x \) and \( v = 0 \) which is a line segment \(-1\) to \(1\) on real axis of \( w \)-plane.

**Case 2:** Image of imaginary axis \( x = 0 \) in the \( z \) plane under \( w = \sin z \)
\( x = 0 \Rightarrow u = 0 \) and \( v = \sinh y \) which is imaginary axis in the \( w \)-plane.

**Case 3:** Image of lines \( x = c \), parallel to the imaginary axis of \( z \) plane under \( w = \sin z \)
\( x = c \Rightarrow u = \sin c \cosh y \) and \( v = \cos c \sinh y \)

Eliminating \( y, \cosh^2 y - \sinh^2 y = 1 \Rightarrow \frac{u^2}{\sinh^2 c} - \frac{v^2}{\cosh^2 c} = 1 \)
which is a hyperbola whose foci are at \((\pm 1, 0)\), with the exception of \( \sin c = 0 \) and \( \cos c = 0 \) \( \Rightarrow \) identify the cases when \( x = \frac{\pi}{2} \) and \( x = -\frac{\pi}{2} \)

**Case 4:** Image of lines \( y = c \), parallel to the real axis of \( z \) plane under \( w = \sin z \)
\( y = c \Rightarrow u = \sin x \cosh c \) and \( v = \cos x \sinh c \)

Eliminating \( x, \Rightarrow \sinh^2 x + \cosh^2 x = 1 \Rightarrow \frac{u^2}{\cosh^2 c} + \frac{v^2}{\sinh^2 c} = 1 \) which is an ellipse whose foci are at \((\pm 1, 0)\).

**Example 21.** Discuss the transformation \( w = z + \frac{1}{z} \)

---

Here we convert the transformation \( w = z + \frac{1}{z} \) into polar coordinates \( z = re^{\theta} \).

Let \( w = z + \frac{1}{z} = u + iv \)
\( z = \frac{1}{2} = e^{\theta} \)
\( \frac{1}{2} = \frac{1}{2} e^{i \theta} + \frac{1}{2} e^{-i \theta} \)
\( = \cos \theta + i \sin \theta + \cos \theta - i \sin \theta \)
\( \Rightarrow u = \cos \theta \) and \( v = -\frac{1}{2} \sin \theta \)

Consider the circles \( |z| = \epsilon \neq 1 \) in the \( z \)-plane.

Then, \( r = \epsilon \Rightarrow u = \epsilon \cos \theta \) and \( v = -\frac{1}{2} \sin \theta \)

Eliminating \( \theta, \frac{u^2}{\epsilon^2} + \frac{v^2}{\epsilon^2} = 1 \)

which is an ellipse in the \( w \)-plane with foci \((\pm 2, 0)\). Thus family of circles \( |z| = \epsilon \neq 1 \) in the \( z \)-plane are mapped into family of confocal ellipses with foci \((\pm 2, 0)\).
8 Linear Fractional Transformation or Mobius transformation

A transformation \( w = \frac{az + b}{cz + d} \) where \( a, b, c \) and \( d \) are constants and \( ad - bc \neq 0 \) is called linear fractional transformation.

NOTE: \( \frac{dw}{dz} = \frac{ad - bc}{(cz + d)^2} \) \( ad - bc \neq 0 \Rightarrow \) conformal for all \( z \)

The following standard transformations are special cases of linear fractional transformation

(i) Translation: \( w = z + a \)

(ii) Rotation: \( w = az \), \( |a| = 1 \)

(iii) Inversion: \( w = \frac{1}{z} \) (inversion in the unit circle)

(iv) Linear Transformation: \( w = az + b \)

Example 22. Discuss the transformation inversion \( w = \frac{1}{z} \)

Let \( z = re^{i\theta} \) and \( w = Re^{i\phi} \), then the transformation becomes \( Re^{i\phi} = \frac{1}{r}e^{-i\theta} \) so that \( R = \frac{1}{r} \) and \( \phi = -\theta \). Thus under the transformation \( w = \frac{1}{z} \), a point \((r, \theta)\) in \( z\)-plane is mapped into the point \((\frac{1}{r}, -\theta)\)
Example 23. Prove that the map $w = \frac{1}{z}$ maps every straight line or circle onto a circle or straight line.

Proof. Every straight line or circle in the $z$-plane can be written

$$A(z^2 + y^2) + Bz + Cy + D = 0$$

($A, B, C, D$ real).

If $A = 0$, then $w = \frac{1}{z}$ is a straight line. If $A \neq 0$, then $z^2 + \frac{y^2}{A} = -\frac{Bz}{A} - \frac{Cy}{A} - \frac{D}{A}$ is a circle.

Now $w = \frac{1}{z}$, substitution of $z = \frac{1}{w}$ and multiplication by $w^2$ give the equation

$$A + \frac{B}{w} + C \frac{1}{w^2} + D \frac{1}{w^3} = 0.$$ 

or, in terms of $x$ and $y$,

$$A + Bz - Cy + Dzw^2 + zw^3 = 0.$$ 

This represents a circle (if $D \neq 0$) or a straight line (if $D = 0$) in the $w$-plane.

Example 24. Every linear fractional transformation $w = \frac{az + b}{cz + d}$ maps the circles and straight lines in $z$ plane onto the circles and straight lines in the $w$-plane.

Every bilinear transformation $w = \frac{az + b}{cz + d}$ is the combination of basic transformations.

(i) translation: $w = z + c$

(ii) rotation and magnification: $w = cz$

(iii) inversion: $w = \frac{1}{z}$

By actual division, we have $w = \frac{az + b}{cz + d} = \frac{bc - ad}{c^2} + \frac{1}{z + d/c}$

Taking $w_1 = \frac{d}{c}$, $w_2 = \frac{1}{w_1}$, $w_3 = \frac{bc - ad}{w_2}$ we get $w = \frac{a}{z} + w_1$

Thus, by these transformations, we successively pass from $z$-plane to $w_1$-plane, from $w_1$-plane to $w_2$-plane and finally from $w_2$-plane to $w$-plane.

Since each of these auxiliary transformations maps circles into circles, hence a bilinear transformation also maps circles into circles.

Definition 8.1. The inverse mapping of linear fractional transformation $w = \frac{az + b}{cz + d}$ is $z = \frac{dw - b}{cw + a}$ which is also a linear fractional transformation.

Definition 8.2. Fixed points of a mapping $w = f(z)$ are points that are mapped onto themselves (kept fixed) i.e. $w = f(z) = z$. 
<table>
<thead>
<tr>
<th>Transformation</th>
<th>Number of Fixed points</th>
</tr>
</thead>
<tbody>
<tr>
<td>Identity map $w = z$</td>
<td>Every points</td>
</tr>
<tr>
<td>$w = z$</td>
<td>Infinitely many</td>
</tr>
<tr>
<td>Inversion $w = \frac{1}{z}$</td>
<td>Two</td>
</tr>
<tr>
<td>Rotation $w = cz$</td>
<td>One</td>
</tr>
<tr>
<td>Translation $w = z + c$</td>
<td>None</td>
</tr>
</tbody>
</table>

**Example 25.** Find the inverse of $w = \frac{z - i}{z + i}$

The inverse mapping of linear fractional transformation $w = \frac{az + b}{cz + d}$ is $z = \frac{dw - b}{cw + a}$

$\Rightarrow$ inverse of $w = \frac{z - i}{z + i}$ is $z = \frac{iw - i}{w + 1}$

**Example 26.** Find the fixed point of the transformation $w = 16z^5$.

Here $f(z) = z^5$

For fixed points $f(z) = z \Rightarrow z^5 = z \Rightarrow z^5 - z = 0 \Rightarrow z(z^4 - 1) = 0$

$\Rightarrow z = 0$ and $z^4 = 1$

$\Rightarrow z = 0$ and $z = \pm 1, \pm i$

$\therefore z = 0, \pm 1$ and $\pm i$ are the fixed points.

**References**

TUTORIAL

1. Find the Maclaurin series of \( \sin\left(\frac{z^2}{2}\right) \) and its radius of convergence.

2. Find the Taylor series of \( f(z) = \frac{1}{z} \) with center at \( z_0 = i \), and find its radius of convergence.

3. Evaluate \( \int_C \text{Re}(z)dz \), where \( C \) is the shortest path from \( 1 + i \) to \( 5 + 5i \).

4. Evaluate \( \int_C \frac{\cos k (z^2 - \pi i)}{z - \pi i} \) \( dz \) where \( c \) is a boundary of a quadrilateral with vertices \( \pm 2, \pm 4i \).

5. Indicate whether Cauchy's integral theorem applies for the integration of \( f(z) \) counter clockwise around the unit circle.
   (a) \( f(z) = \frac{1}{4z - 3} \)  
   (b) \( f(z) = z^3 \cot z \)

ASSIGNMENT

1. Find the Maclaurin series of \( \frac{z + 2}{1 - z^2} \) and its radius of convergence.

2. Find the Taylor series of \( f(z) = \frac{1}{1 + z} \) with center at \( z_0 = -i \), and find its radius of convergence.

3. Evaluate \( \int_C \text{Re}(z)dz \), where \( C \) is the parabola \( y = 1 + \frac{1}{2}(x - 1)^2 \) from \( 1 + i \) to \( 3 + 3i \).

4. Evaluate \( \int_C \text{Im}(z^2)dz \) counterclockwise around the triangle with vertices \( 0, 1, i \).

5. Evaluate \( \int_C \sec^2 z \) \( dz \), where \( C \) is any path from \( \frac{\pi}{4} \) to \( \frac{\pi i}{4} \).

6. Evaluate \( \int_C \tan zdz \) where \( C \) is \( |z| = 2 \).

7. Evaluate \( \int_C \frac{e^z}{z^3 - (z + i)^3} \) \( dz \) where \( C \) is the circle \( |z| = 2 \).

8. Evaluate \( \int_C \frac{e^{\pi z}}{\cos \pi z} \) \( dz \) where \( C \) is \( |z| = 1 \)

9. Evaluate \( \int_C \frac{z + 4}{z^2 + 2z + 5} \) \( dz \) where \( C \) is \( |z + 1 - i| = 2 \)

10. Evaluate \( \int_C \frac{\sin (\pi z^2) + \cos (\pi z^2)}{(z - 1)(z - 2)} \) \( dz \) where \( C \) is \( |z| = 3 \)
UNIT-WISE QUESTION BANK

1. Find the Maclaurin series of $\sin^2(z)$ and its radius of convergence.

2. Find the Taylor series of $f(z) = \cos z$ with center at $z_0 = \pi$, and find its radius of convergence.

3. Find the Taylor series of $f(z) = \frac{1}{(z - i)^2}$ with center at $z_0 = -i$, and find its radius of convergence.

4. Evaluate $\int_C e^z dz$, where $C$ is the shortest path from $\frac{\pi}{2i}$ to $\pi i$.

5. Evaluate $\int_C z e^{z^2} dz$, where $C$ is the path from 1 along the axes to $i$.

6. Evaluate $\int_C Re(z^2) dz$ clockwise around the boundary of the square with vertices $0, i, 1 + i, 1$.

7. Show that $\int_C \frac{1}{z} dz = \pi i$ or $-\pi i$ according as $C$ is the semicircle $|z| = 1$ above or below the real axis from $(1, 0)$ to $(-1, 0)$.

8. Evaluate $\int_C \frac{\cos \pi z}{z^2 - 1} dz$ where $C$ is the rectangle with vertices $2 \pm i, -2 \pm i$.

9. Evaluate $\int_C \frac{e^z}{(z + 1)^3} dz$ where $C$ is $|z + 1| = 1$.

10. Evaluate $\int_C \frac{z^2 + 5z + 3}{(z - 2)^3} dz$ where $C$ is $|z| = 3$.

11. Using Cauchy’s integral formula, Evaluate $\int_C \frac{z^2}{(z - 1)^2(z + 2)} dz$ where $C$ is $|z - 2| = 2$.

12. Evaluate $\int_C \frac{\cos \pi z^2}{(z - 1)(z - 2)} dz$ where $C$ is $|z| = 3$.

13. Evaluate $\int_C \frac{2z - 1}{z(z + 1)(z - 1)} dz$ where $C$ is $|z| = 2$.

14. Evaluate $\int_C \frac{dz}{z^2(z - 1)} dz$ where $C$ is $|z| = 2$.

15. Evaluate $\int_C \frac{2z - 1}{z(z + 1)(z - 1)} dz$ where $C$ is $|z| = 2$.

16. Evaluate $\int_C \frac{z e^z}{(z + 1)(z + 2)} dz$ where $C$ is $|z - i| = 3$.
Course Handout
MAT 201 : Partial Differential Equations and Complex Analysis
Module 4 : Complex Variable - Integration

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Pre requisites

(i) Complex valued function

(ii) Analytic function

(iii) Properties of Analytic function

(iv) Partial fraction decomposition

(v) Taylor and Maclaurin series in the real case

Complex integration

Some real integrals appearing in applications cannot be evaluated easily using usual methods. Complex integration is useful because it allows us to evaluate such real integrals.

As in the real case, we have complex indefinite integrals as well as complex definite integrals (also called line integrals).

1 Line integrals in the complex plane

Definition 1.1. An indefinite integral is a function whose derivative equals a given analytic function.

Definition 1.2. Complex definite integrals are called line integrals.

They are denoted by

\[ \int_C f(z)dz, \oint_C f(z)dz. \]

Here \( f(z) \) is integrated over the given curve \( C \) or a portion of it. This curve \( C \) is called the path of integration.

Definition 1.3. \( C \) may be represented by a parametric representation

\[ z(t) = x(t) + iy(t), \quad (a \leq t \leq b) \]
The sense of increasing $t$ is called the positive sense on $C$, and we say that $C$ is oriented by the above parametrisation.

**Example 1.** $z(t) = t + i3t$, $(0 \leq t \leq 2)$ gives the portion of the line $y = 3x$ between $(0,0)$ and $(2,6)$, oriented in the direction from $(0,0)$ to $(2,6)$.

**Example 2.** $z(t) = 4 \cos t + 4i \sin t$ ($-\pi \leq t \leq \pi$) represents the circle $|z| = 4$ oriented in the counterclockwise direction.

### 1.1 Basic properties

(i) **Linearity:**

\[
\int_C [k_1 f_1(z) + k_2 f_2(z)] dz = k_1 \int_C f_1(z) dz + k_2 \int_C f_2(z) dz
\]

(ii) **Reversing the sense while integrating over the same path introduces a minus sign:**

\[
\int_{-\pi}^{\pi} f(z) dz = -\int_{\pi}^{-\pi} f(z) dz
\]

(iii) **Partitioning of path:**

\[
\int_C f(z) dz = \int_{C_1} f(z) dz + \int_{C_2} f(z) dz
\]

### 2 First evaluation method - indefinite integration and substitution of limit

This method can be applied only for analytic functions.

Let $f(z)$ be analytic in a simply connected domain $D$. Then there exists an indefinite integral of $f(z)$ in $D$, that is, an analytic function
$F(z)$ such that $F'(z) = f(z)$ and for all paths in $D$ joining $a$ and $b$, we have

$$\int_a^b f(z)dz = F(b) - F(a)$$

Example 3. $\int_0^{1+i} z^2dz = \frac{z^3}{3}\bigg|_0^{1+i} = \frac{(1+i)^3}{3} = -\frac{2}{3} + \frac{2}{3}i$

Example 4. $\int_{-\pi i}^{\pi i} \cos zdz = \sin z\bigg|_{-\pi i}^{\pi i} = 2\sin(\pi i)$

Example 5. $\int_{8+3\pi i}^{8-3\pi i} e^{z/2}dz = 2e^{8/2}\bigg|_{8+3\pi i}^{8-3\pi i} = \left[e^{4-\frac{3\pi}{2}i} - e^{4+\frac{3\pi}{2}i}\right]$

3 Second evaluation method - use of a representation of a path

This method can be applied to any continuous complex functions, not just analytic functions.

This method uses a parametric representation of a path.

Let $C$ be a piecewise smooth path with parametric representation $z = z(t)$, where $a \leq t \leq b$.

Let $f(z)$ be a continuous function on $C$. Then

$$\int_C f(z)dz = \int_a^b f(z(t))z'(t)dt$$

Steps in second evaluation method

(i) Represent $C$ in the form $z(t) = x(t) + iy(t) \ [a \leq t \leq b]$.

(ii) Calculate $z'(t) = dz/dt$

(iii) Substitute $z(t)$ for $z$ in $f(z)$.

(iv) Integrate $f(z(t))z'(t)$ over $t$ from $a$ to $b$. 
Example 6. Evaluate \( \int_C \text{Re} \, dz \), \( C \) is the shortest path from 0 to \( 1 + i \).

Solution:

If \( z = x + iy \), \( f(z) = \text{Re}(z) = x \). This function doesn’t satisfy the Cauchy-Riemann equations; it is not analytic.

So, we will apply the second evaluation method.

(i) The shortest path from 0 to \( 1 + i \) is given by the segment of \( y = x \) from \((0, 0)\) to \((1, 1)\).

This gives us the required parametric representation:

\[ z = x(t) + iy(t) = t + it \quad (0 \leq t \leq 1) \]

Here \( x(t) = y(t) = t, a = 0, b = 1 \)

(ii) \( z'(t) = 1 + i \)

(iii) \( f(z(t)) = x(t) = t \)

(iv)

\[ \int_C f(z)dz = \int_a^b f(z(t))z'(t)dt = \int_0^1 t(1 + i)dt = \frac{1 + i}{2} \]

Example 7. Evaluate \( \int_C \text{Re} \, dz \), \( C \) is the part of \( y = x^2 \) from 0 to \( 1 + i \).

Solution:

If \( z = x + iy \), \( f(z) = \text{Re}(z) = x \). This function doesn’t satisfy the Cauchy-Riemann equations; it is not analytic.

So, we will apply the second evaluation method.

(i) The given path \( C \) from 0 to \( 1 + i \) is the portion of \( y = x^2 \) from \((0, 0)\) to \((1, 1)\).

This gives us the required parametric representation:

\[ z = x(t) + iy(t) = t + it^2 \quad (0 \leq t \leq 1) \]
Here \( x(t) = t \), \( y(t) = t^2, a = 0, b = 1 \)

(ii) \( z'(t) = 1 + i2t \)

(iii) \( f(z(t)) = x(t) = t \)

(iv)

\[
\int_{C} f(z)dz = \int_{a}^{b} f(z(t))z'(t)dt = \int_{0}^{1} t(1 + i2t)dt = \frac{1}{2} + \frac{2}{3}i
\]

(The following examples derive 2 very important results in complex analysis)

Example 8. Evaluate \( \oint_{C} \frac{dz}{z} \), where \( C \) is the unit circle (circle of radius 1 and center 0), oriented counterclockwise.

Solution:

(i) Using polar coordinates, a point \((x, y)\) on a circle of radius 1 can be represented as \( x = 1 \cos t, y = 1 \sin t \).

This gives us the parametric representation \( z(t) = \cos t + i \sin t = e^{it} \) (applying Euler’s formula).

As the unit circle is oriented counterclockwise, as \( t \) increases, we should move in a counterclockwise direction along the circle.

This gives the variation of \( t \) as \( 0 \leq t \leq 2\pi \).

(ii) \( z'(t) = ie^{it} \), applying Chain rule.

(iii) \( f(z(t)) = \frac{1}{z(t)} = \frac{1}{e^{it}} = e^{-it} \)

(iv)

\[
\oint_{C} f(z)dz = \int_{a}^{b} f(z(t))z'(t)dt = \int_{0}^{2\pi} e^{-it}ie^{it}dt = i \int_{0}^{2\pi} dt = 2\pi i
\]
Example 9. Evaluate \( \oint_C f(z)dz \), given \( f(z) = (z - z_0)^m \), where \( m \) is an integer and \( z_0 \) is a complex constant; \( C \) is the circle with radius \( \rho \) and center at \( z_0 \).

Solution:

(i) The parametrisation of a circle with radius \( \rho \) and center at \( z_0 \) is given by \( z(t) = z_0 + \rho(\cos t + i \sin t) = z_0 + \rho e^{it} \) \((0 \leq t \leq 2\pi)\)

(ii) \( z'(t) = i\rho e^{it} \), applying Chain rule.

(iii) \( f(z(t)) = (z(t) - z_0)^m = \rho^m e^{imt} \)

(iv) \[
\oint_C f(z)dz = \int_a^b f(z(t))z'(t)dt = \int_0^{2\pi} \rho^m e^{imt} i\rho e^{it} dt
\]

\[
= i\rho^{m+1} \int_0^{2\pi} e^{i(m+1)t} dt
\]

Applying Euler’s formula,
\[
\oint_C (z - z_0)^m dz = i\rho^{m+1} \int_0^{2\pi} [\cos(m + 1)t + i\sin(m + 1)t] dt
\]

To evaluate this integral, we consider 2 cases:

(a) \( m = -1 \)

Letting \( m = -1 \), \( \rho^{m+1} = 1 \), \( \cos(m + 1)t = 1 \), \( \sin(m + 1)t = 0 \).

For this case,
\[
\oint_C (z - z_0)^m dz = 2\pi i
\]

(b) \( m \neq -1 \)

Here
\[
\oint_C (z - z_0)^m dz = i\rho^{m+1} \left| \left. \frac{\sin(m + 1)t}{m + 1} - i \frac{\cos(m + 1)t}{m + 1} \right| \right|_0^{2\pi}
\]
4.1 Cauchy’s Integral Theorem

**Statement**
If \( f(z) \) is analytic in a simple connected domain \( D \), then for every simple closed path \( C \) in \( D \), \( \oint_C f(z) \, dz = 0 \).

![Diagram of a simple connected domain and a simple closed path](image)

**Example 10.** For any closed path \( C \), since these functions are analytic \( \forall z \)

(i) \( \oint_C e^z \, dz = 0 \)

(ii) \( \oint_C \cos z \, dz = 0 \)

(iii) \( \oint_C z^n \, dz = 0 \) for \( n = 0, 1, \ldots \)

(iv) \( \oint_C \frac{1}{2z - 1} \neq 0 \) for \( |c| : |z| = 1 \)

(v) \( \oint_C \bar{z} \, dz = \int_0^{2\pi} e^{-it}i e^{it} \, dt = 2\pi i \neq 0 \) where \( C : z(t) = e^{it} \) is a unit circle.
4.2 Cauchy's Integral Formula

Statement 1.
If $f(z)$ is analytic in a simple connected domain $D$, then for any point $z_0$ in $D$ and any simple closed path $C$ in $D$ that encloses $z_0$,

$$\oint_c \frac{f(z)}{z - z_0} \, dz = 2\pi i f(z_0)$$

the integration being taken counterclockwise.

Definition 4.4. Statement 2
If $f(z)$ is analytic in a domain $D$, then it has derivatives of all orders in $D$ which are then also analytic functions in $D$. The values of these derivatives at a point $z_0$ in $D$ are given by the formula

$$f'(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^2} \, dz$$

and in general

$$f^n(z_0) = \frac{n!}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^{n+1}} \, dz$$

here $C$ is any simple closed path in $D$ that encloses $z_0$ and whose full interior belongs to $D$.

Example 11. Evaluate $\oint_C \frac{\cos \pi z}{(z - \pi i)^2} \, dz$ for any contour $C$ enclosing the point $\pi i$.

Solution By Cauchy's Integral Formula,
\[ \int_C \frac{\cos \frac{\pi z}{2}}{(z - \pi i)^3} \, dz = 2\pi i (\cos z) \bigg|_{\pi i} = -2\pi i \sin \pi i = 2\pi \sinh \pi. \]

**Example 12.** For any contour enclosing the point \(-i\), find
\[ \int_C \frac{z^4 - 3z^2 + 6}{(z + i)^3} \, dz \]

**Solution** By Cauchy’s Integral formula,
\[ \int_C \frac{z^4 - 3z^2 + 6}{(z + i)^3} \, dz = \frac{2\pi i}{2!} \left( z^4 - 3z^2 + 6 \right) \bigg|_{z = -i} = \pi i [12z^2 - 6]_{z = -i} = -18\pi i \]

**Example 13.** For any contour \( C \), 1 lies inside and \( \pm 2i \) lie outside, Evaluate \( \int_C \frac{e^z}{z - 1)^2(z^2 + 4)} \, dz \)

**Solution** By Cauchy’s Integral formula,
\[ \int_C \frac{e^z}{(z - 1)^2(z^2 + 4)} \, dz = \int_C \frac{e^z}{(z - 1)^2} - \int_C \frac{e^z}{(z^2 + 4)} \, dz \]
\[ = 2\pi i \frac{e^z(z^2 + 4) - e^z 2z}{(z^2 + 4)^2} \bigg|_{z = 1} = \frac{6pe_i}{25} \]

**Example 14.** Evaluate \( \int_C \frac{\sin z}{z^4} \, dz \) where \( |C| : |z| = 1 \)

**Solution** By Cauchy’s Integral formula,
\[ \int_C \frac{\sin z}{z^4} \, dz = \int_C \frac{2\pi i \sin z}{3!} \, dz \bigg|_{z = 0} = \frac{\pi i}{3} \left( -\cos z \right) \bigg|_{z = 0} = -\frac{\pi}{3} \]

5 Taylor and Maclaurin series

**Definition 5.1.** The **Taylor series** of a function \( f(z) \) centered at \( z_0 \) is given by
\[ f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n \]
where

\[ a_n = \frac{1}{n!} f^{(n)}(z_0) \]

If \( z_0 = 0 \), this series is called the **Maclaurin series**.

### 5.1 Important Special Taylor Series

(i) **Geometric Series**:

\[ \frac{1}{1 - z} = \sum_{n=0}^{\infty} z^n = 1 + z + z^2 + \cdots \quad \text{converges when } |z| < 1 \]

(ii) **Binomial series**:

\[ \frac{1}{(1 + z)^m} = \sum_{n=0}^{\infty} \binom{-m}{n} z^n = 1 - mz + \frac{m(m+1)}{2!}z^2 - \frac{m(m+1)(m+2)}{3!}z^3 + \cdots \]

converges when \( |z| < 1 \)

(iii) **Exponential function**:

\[ e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} = 1 + z + \frac{z^2}{2!} + \cdots \]

(iv) **Trigonometric functions**:

\[
\cos z = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} z^{2n} = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} - \cdots
\]

\[
\sin z = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{2n+1} = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \cdots
\]

(v) **Hyperbolic functions**:

\[
\cosh z = \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!} = 1 + \frac{z^2}{2!} + \frac{z^4}{4!} + \cdots
\]
\[
\sinh z = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!} = z + \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots
\]

(vi) **Logarithm:**

\[
\ln(1 + z) = z - \frac{z^2}{2} + \frac{z^3}{3} - \cdots
\]

In practice, evaluating the Taylor series of a function directly using the definition is difficult. So we often use known Taylor series to compute the Taylor series for a given function.

**Example 15.** Find the Maclaurin series of \( f(z) = \frac{1}{1+z^2} \).

The given function is similar to the sum of the geometric series.

\[
f(z) = \frac{1}{1+z^2} = \frac{1}{1-(-z^2)} = \sum_{n=0}^{\infty} (-z^2)^n = 1 - z^2 + z^4 - z^6 + \cdots
\]

As the geometric series converges when its common ratio has magnitude less than 1, the above series converges when \(|z^2| < 1\), that is, when \(|z| < 1\).

**Example 16.** Find the Maclaurin series of \( f(z) = \tan^{-1} z \).

If we differentiate the given function, from the previous question, we have

\[
f'(z) = \frac{1}{1+z^2} = 1 - z^2 + z^4 - z^6 + \cdots
\]

Now integrate both sides of the above equation

\[
f(z) = z - \frac{z^3}{3} + \frac{z^5}{5} - \frac{z^7}{7} + \cdots
\]

**Example 17.** Find the Taylor series of \( \frac{1}{e^{-z}} \) centered at \( z_0 \).

The Taylor series centered at \( z_0 \) will contain powers of \( z - z_0 \). To get those terms, add and subtract \( z_0 \) in the denominator.
\[ \frac{1}{c - z} = \frac{1}{c - z_0 + z_0 - z} = \frac{1}{(c - z_0) - (z - z_0)} \]

Now use geometric series.

\[ = \frac{1}{(c - z_0) \left(1 - \frac{z - z_0}{c - z_0}\right)} = \frac{1}{(c - z_0)} \sum_{n=0}^{\infty} \left(\frac{z - z_0}{c - z_0}\right)^n \]

This series converges when \[\left|\frac{z - z_0}{c - z_0}\right| < 1\], that is, when \[|z - z_0| < |c - z_0|\].

**Example 18.** Develop the following function in powers of \(z - 1\).

\[ f(z) = \frac{2z^2 + 9z + 5}{z^3 + z^2 - 8z - 12} \]

Decomposing the given function using partial fractions,

\[ f(z) = \frac{1}{(z + 2)^2} + \frac{2}{z - 3} \]

To get powers of \(z - 1\), add and subtract 1 in both the denominators.

\[ = \frac{1}{(z - 1 + 1)^2} + \frac{2}{z - 1 + 1 - 3} \]

\[ = \frac{1}{(3 + (z - 1))^2} - \frac{2}{2 - (z - 1)} \]

Now use binomial and geometric series.

\[ = \frac{1}{9} \left[ \left(1 + \frac{z - 1}{3}\right)^2 \right] - \frac{1}{1 - \left(\frac{z - 1}{2}\right)} \]

\[ = \frac{1}{9} \sum_{n=0}^{\infty} \left(-2\right)^n \left(\frac{z - 1}{3}\right)^n - \sum_{n=0}^{\infty} \left(\frac{z - 1}{2}\right)^n \]

The first series in the above sum converges when \[\left|\frac{z - 1}{3}\right| < 1\], or \[|z - 1| < 3\].
The second series in the above sum converges when \( \frac{|z-1|}{2} < 1 \), or \( |z - 1| < 2 \).

Hence the sum of the above 2 series converges in the region where both the series converge. This region is \( |z - 1| < 2 \).

References

MAT 201

RAJAGIRI SCHOOL OF ENGINEERING & TECHNOLOGY
DEPARTMENT OF MATHEMATICS
TUTORIAL / ASSIGNMENT/ UNIT-WISE QUESTION BANK RECORD BOOK
COURSE:- MAT 201; PARTIAL DIFFERENTIAL EQUATIONS AND COMPLEX ANALYSIS
Branch: COMMON FOR ALL BRANCHES (EXCEPT CS & IT)

MODULE V COMPLEX VARIABLE - RESIDUE INTEGRATION

TUTORIAL

1. Find the Laurent series expansion of $\frac{\cos z}{z^4}$ at the singular point.

2. Find the Laurent series expansion of $\frac{\sinh 2z}{z^2}$ at the singular point $z = 0$.

3. Find all singular points and the corresponding residues:

(a) $\frac{1}{4 + z^2}$

(b) $\frac{\sin z}{z^6}$

4. Evaluate $\int_{|z|=1} e^{1/z} \, dz$

5. Evaluate $\int_{0}^{2\pi} \frac{d\theta}{7 + 6\cos \theta}$

ASSIGNMENT

1. (a) Find the Laurent series expansion of $\frac{\cos z}{(z - \pi)^2}$ at the singular point $z = \pi$.

(b) Find the Laurent series expansion of $\frac{e^{\alpha z}}{z - b}$ at the singular point $z = b$.

2. Expand the following function in Laurent’s series

(a) $\frac{1}{z^4 - 4z + 3}$ for $1 < |z| < 3$

(b) $\frac{1}{z(z - 1)(z - 2)}$ for $|z| > 2$

3. Determine singularities of the following function and classify

(a) $\frac{\sin z^2}{z}$

(b) $\frac{\cos z - 1}{z^2}$

(c) $\frac{1 - e^{2z}}{z^4}$

4. Find the zeros and its order of the following function

(a) $(1 - z)^2$
5. Find all singular points and the corresponding residues:

   (a) \( \frac{z^2 + 1}{z^3 - z} \)
   (b) \( \frac{\cos z}{z^6} \)
   (c) \( \sec z \)

6. Evaluate \( \int_{|z|=1} \tan \pi z dz \).

7. Evaluate \( \int_{|z|=1} \frac{1 - 4z + 6z^2}{(z^2 + \frac{1}{4})(2 - z)} dz \).

8. Evaluate \( \int_0^{2\pi} \frac{\cos \theta}{7 + 6 \cos \theta} d\theta \).

9. Evaluate \( \int_{-\infty}^{\infty} \frac{dx}{x^2 + 1} \).

10. Evaluate \( \int_{-\infty}^{\infty} \frac{\sin x}{x^2 + 1} dx \).

**UNIT-WISE QUESTION BANK**

1. Find the Laurent series expansion of \( \frac{1}{z(z - i)} \) at the singular point \( z = i \).

2. Find the Laurent series expansion of \( z^3 \cosh \frac{1}{z} \) at the singular point \( z = 0 \).

3. Determine singularities of the following function
   (a) \( \tan \pi z \)
   (b) \( \cot z \)
   (c) \( \frac{1}{1 - e^z} \)

4. Expand the following function in Laurent’s series
   (a) \( \frac{1}{z - 2} \), for \( |z| > 2 \)
   (b) \( \frac{z^2 - 1}{(z + 2)(z + 3)} \), for \( |z| > 3 \)

5. Find the Laurent series expansion of \( \frac{1 - \cos z}{z^3} \), about \( z = 0 \).

6. Find all singular points and the corresponding residues:
   (a) \( \frac{1}{(z^2 - 1)^2} \)
   (b) \( \frac{1/3}{z^4 - 1} \)
   (c) \( \frac{z^2}{z^4 - 1} \)
7. Evaluate $\int_{|z|=2} \tan \pi z \, dz$.

8. Evaluate $\int_{|z|=4.5} \frac{e^z}{\cos z} \, dz$.

9. Evaluate $\int_{|z|=1} \frac{\cosh z}{z^3 - 3iz} \, dz$.

10. Evaluate $\int_{0}^{2\pi} \frac{1 + 4 \cos \theta}{17 - 8 \cos \theta} \, d\theta$.

11. Evaluate $\int_{0}^{2\pi} \frac{1}{5 - 4 \sin \theta} \, d\theta$.

12. Evaluate $\int_{0}^{2\pi} \frac{\sin^3 \theta}{5 - 4 \cos \theta} \, d\theta$.

13. Evaluate $\int_{-\infty}^{\infty} \frac{dx}{(x^2 - 2x + 5)^2}$.

14. Evaluate $\int_{-\infty}^{\infty} \frac{\cos x}{x^4 + 1} \, dx$.

15. Evaluate $\int_{-\infty}^{\infty} \frac{\sin 3x}{x^4 + 1} \, dx$. 
Course Handout
MAT 201 : Partial Differential Equations and Complex Analysis
Module 5 : COMPLEX VARIABLE-RESIDUE INTEGRATION

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Pre requisites

(i) Cauchys Integral Theorem

(ii) Cauchys Integral Formula

(iii) Power series representation of analytic function-Taylor Series
1 Laurent’s Series

Laurent series generalize Taylor series. If we want to develop a function \( f(z) \) in powers of \( z - z_0 \) when \( f(z) \) is singular at \( z_0 \) (we cannot use a Taylor series), then we can use a new kind of series called Laurent’s Series.

Let \( f(z) \) be analytic in a domain containing two concentric circles and with center \( z_0 \) and the annulus between them. Then \( f(z) \) can be represented by the Laurent series

\[
 f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n}
\]

\[ = a_0 + a_1 (z - z_0) + a_2 (z - z_0)^2 + \ldots + b_1 + \frac{b_2}{(z - z_0)} + \frac{b_3}{(z - z_0)^2} + \ldots \]

consisting of non-negative and negative powers. The co-efficient of Laurent’s series can be defined are given by the integrals

\[
a_n = \frac{1}{2\pi i} \oint_C \frac{f(z^*)}{(z^* - z_0)^{n+1}} dz^*, \quad b_n = \frac{1}{2\pi i} \oint_C \frac{f(z^*)}{(z^* - z_0)^{-(n+1)}} dz^*
\]

taken counterclockwise around any closed path \( C \) that lies in the annulus and encircles in inner circle.

In this case the series (or finite sum) of the negative powers is called the principal part of \( f(z) \) at \( z_0 \) (or of that Laurent series).

Example 1. Find the Laurent’s series of \( z^{-5} \sin z \) with center 0
\[ f(z) = z^{-5} \sin z = \frac{\sin z}{z^5} \]

By Maclaurin’s series \( \sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \ldots \) \((|z| > 0)\)

\[ f(z) = \frac{\sin z}{z^5} \]
\[ = \frac{1}{z^4} \left[ z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \ldots \right] \]
\[ = \frac{1}{z^4} - \frac{1}{3! \cdot z^2} + \frac{1}{5!} - \frac{z^2}{7!} + \ldots \]

**Example 2. Find the Laurent’s series of** \( z^2 e^{1/z} \) **with center 0**

By Maclaurin’s series \( e^t = 1 + \frac{t}{1!} + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \ldots \) \((|t| > 0)\)

\[ f(z) = z^2 e^{1/z} \]
\[ = z^2 \left[ 1 + \frac{1}{z} + \frac{1}{2! \cdot z^2} + \frac{1}{3! \cdot z^3} + \frac{1}{4! \cdot z^4} + \ldots \right] \]
\[ = z^2 + z + \frac{1}{2!} + \frac{1}{3! \cdot z} + \frac{1}{4! \cdot z^2} + \ldots \]

**Example 3. Find Laurent series of** \( \frac{1}{z^3 - z^4} \) **with center zero**

\[ f(z) = \frac{1}{z^3 - z^4} \]
\[ = \frac{1}{z^3(1 - z)} \]
\[ = \frac{1}{z^3} (1 - z)^{-1} \]
\[ = \frac{1}{z^3} (1 + z + z^2 + z^3 + \ldots) \] \((|z| > 0)\)
\[ = \frac{1}{z^4} + \frac{1}{z^3} + \frac{1}{z^2} + 1 + z + z^2 + \ldots \]

**Example 4. Find the Laurent’s series expansion of** \( \frac{e^z}{(z-1)^2} \) **with center} z_0 = 1.**
Put $z - 1 = t \Rightarrow z = t + 1$

$$f(z) = \frac{e^z}{(z-1)^2}$$

$$= \frac{e^{t+1}}{t^2}$$

$$= e^{t} e^{-1}$$

$$= \frac{e}{t^2} \left[ 1 + \frac{t}{2!} + \frac{t^2}{3!} + \frac{t^3}{4!} + \ldots \right] \quad (|t| > 0)$$

$$= e \left[ \frac{1}{t^2} + \frac{1}{t} + \frac{1}{2!} + \frac{t}{3!} + \frac{t^2}{4!} + \ldots \right]$$

$$= e \left[ \frac{1}{(z-1)^2} + \frac{1}{(z-1)} + \frac{1}{2!} + \frac{(z-1)}{3!} + \frac{(z-1)^2}{4!} + \ldots \right] \quad (|z-1| > 0)$$

Example 5. Using Laurent’s series expand $\frac{1}{z^3 - 3z + 2}$ in the region

(i) $|z| < 1$

(ii) $1 < |z| < 2$

(iii) $|z| > 1$

(iv) $0 < |z-1| < 1$

Here $f(z) = \frac{1}{z^3 - 3z + 2} = \frac{1}{(z-1)(z-2)} = \frac{1}{z-2} - \frac{1}{z-1}$ (By partial fractions)

(i) $|z| < 1 \Rightarrow \frac{|z|}{2} < 1$

$$f(z) = \frac{1}{z-2} - \frac{1}{z-1}$$

$$= \frac{1}{2} \left( (1 - z)^{-1} + (1 - z)^{-1} \right)$$

$$= \frac{1}{2} \left( 1 + \frac{z}{2} + \frac{z^2}{4} + \ldots \right) + (1 + z + z^2 + z^3 + \ldots)$$

$$= \frac{z}{2} + \frac{3z^2}{4} + \frac{7z^3}{8} + \frac{15z^4}{16} + \ldots$$
(ii) $1 < |z| < 2 \Rightarrow \frac{1}{|z|} < 1$ and $\frac{|z|}{2} < 1$

$$f(z) = \frac{1}{-2(1 - \frac{z}{2})} - \frac{1}{z(1 - \frac{1}{z})}$$

$$= -\frac{1}{2} \left(1 - \frac{z}{2}\right)^{-1} - \frac{1}{z} \left(1 - \frac{1}{z}\right)^{-1}$$

$$= -\frac{1}{2} \left(1 + \frac{z}{2} + \frac{z^2}{4} + \ldots\right) - \frac{1}{z} \left(1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \ldots\right)$$

$$= -\frac{1}{2} \left(1 + \frac{z}{2} + \frac{z^2}{4} + \ldots\right) - \frac{1}{z} \left(1 + \frac{1}{z^3} + \frac{1}{z^4} + \ldots\right)$$

(iii) $|z| > 1 \Rightarrow 1 < |z| \Rightarrow \frac{1}{|z|} < 1$ (refer previous cases)

(iv) $0 < |z - 1| < 1$

$$f(z) = \frac{1}{z - 2} - \frac{1}{z - 1}$$

$$= \frac{1}{z - 2} - \frac{1}{z - 1}$$

$$= -\frac{1}{(z - 1 - 1)} - \frac{1}{z - 1}$$

$$= -(1 - (z - 1))^{-1} - \frac{1}{z - 1}$$

$$= - (1 + (z - 1) + (z - 1)^2 + (z - 1)^3 + \ldots) - \frac{1}{z - 1}$$

2 **Singularities and Zeros**

**Definition 2.1.** A function $f(z)$ is singular or has a singularity at a point $z = z_0$ if $f(z)$ is not analytic at $z = z_0$.

No other singularities in the neighborhood of singularity $z_0$ of $f(z)$, then $z_0$ is called an **Isolated singularity**, otherwise it is called non-isolated singularity.

**Example 6.** $\tan z$ has an isolated singularity at $z = \pm \frac{\pi}{2}, \pm \frac{3\pi}{2}, \ldots$ and $\tan \frac{1}{z}$ has an **non-isolated singularity** at $z = 0$.

Isolated singularities of $f(z)$ at $z = z_0$ can be classified by Laurent’s series.
(i) Removable singularity:- No negative power terms in Laurent’s series expansion

Example 7. \( \frac{\sin z}{z} = \frac{1}{z} \left( z - \frac{z^3}{3!} + \frac{z^5}{5!} - \ldots \right) = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \ldots \)

\( \therefore \frac{\sin z}{z} \) has removable singular point at \( z = 0 \)

(ii) Isolated essential singularity:- Infinite number of negative power terms in Laurent’s series expansion

Example 8. \( f(z) = \frac{1}{z} \left( 1 + \frac{1}{z^2} + \frac{1}{2z^3} + \frac{1}{3!z^4} + \ldots \right) \) at \( z = 0 \) is an isolated essential singularity.

(iii) Pole type singularity-Finite number of negative power terms in Laurent’s series expansion

Example 9. (a)

\[
\begin{align*}
    f(z) &= \frac{\sin z}{z^5} \\
         &= \frac{1}{z^5} \left( z - \frac{z^3}{3!} + \frac{z^5}{5!} - \ldots \right) \\
         &= \frac{1}{z^4} - \frac{1}{3!}z^2 + \frac{1}{5!} - \frac{z^2}{7!} + \ldots \\
\end{align*}
\]

\( \therefore z = 0 \) is a pole type singularity of order 4

(b) \( f(z) = \frac{1}{z(z-2)^2} \), \( z = 0 \) is a simple pole and \( z = 2 \) is a pole of order 5.

2.1 Zeros of Analytic Function

Definition 2.2. A zero of an analytic function \( f(z) \) in a domain \( D \) is \( z = z_0 \) in \( D \) such that \( f(z_0) = 0 \). A zero \( z_0 \) of \( f(z) \) has order \( n \) if \( f(z_0) = 0 \), \( f'(z_0) = 0 \), \( f''(z_0) = 0 \), \( \ldots \), \( f^{n-1}(z_0) = 0 \) and \( f^{n}(z) \neq 0 \).

Example 10. \( f(z) = \sin(z^2) \), \( z = 0 \) is a zero at \( z = 0 \)

\[
\begin{align*}
    f(z) &= \sin(z^2) \\
    f'(z) &= \cos(z^2)(2z) \\
    f''(z) &= 2(\cos(z^2) - z\sin(z^2)2z) \\
\end{align*}
\]

\( \therefore f(z) \) has zero at \( z = 0 \) of order 2.

Example 11. \( f(z) = 1 - \cos z \) has second order zeros at \( z = 0, \pm 2\pi, \pm 4\pi, \ldots \)
3 Residue Integration

Some real integrals appearing in applications cannot be evaluated easily using usual methods. Complex integration is useful because it allows us to evaluate such real integrals.

The method we will use to evaluate such integrals is called residue integration.

Let \( f(z) \) be singular at \( z_0 \) but otherwise analytic. Then \( f(z) \) has a Laurent series centered at \( z_0 \) given by

\[
f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n + \frac{b_1}{z - z_0} + \frac{b_2}{(z - z_0)^2} + \cdots
\]

**Definition 3.1.** The term \( b_1 \) in the Laurent series expansion of \( f(z) \) is called the **Residue of \( f(z) \) at** \( z = z_0 \).

It is denoted by \( b_1 = \text{Res}_{z=z_0} f(z) \)

---

**Example 12.** Evaluate \( \int_{|z|=1/2} e^{1/z} \, dz \).

**Solution:**

(i) The function to be integrated \( f(z) = e^{1/z} \) has a singularity at \( z = 0 \).

(ii) \( z = 0 \) is inside the given contour, so we can apply the above result.

(iii) The Laurent series of \( e^{1/z} \) centered at \( z = 0 \) is

\[
e^{1/z} = 1 + \frac{1}{1!z} + \frac{1}{2!z^2} + \cdots
\]

(iv) \( \text{Res}_{z=0} e^{1/z} = \frac{1}{1!} = 1 \)

(v) \( \int_{|z|=1/2} e^{1/z} \, dz = 2\pi i \text{Res}_{z=0} e^{1/z} = 2\pi i \)
Example 13. Integrate $f(z) = \frac{\sin z}{z^4}$ anticlockwise around the unit circle.

Solution:

(i) $f(z)$ has a singularity at $z = 0$.

(ii) $z = 1$ is inside the contour $|z| = 1$.

(iii) The Laurent series of $f(z)$ centered at $z = 0$ is

$$\frac{\sin z}{z^4} = \frac{1}{z^4} - \frac{1}{3!z^3} + \frac{z^2}{5!} - \cdots$$

(iv) $\text{Res}_{z=0} f(z) = -\frac{1}{3!}$

(v) $\oint_{|z|=1} \frac{\sin z}{z^4} dz = 2\pi i \text{Res}_{z=0} \frac{\sin z}{z^4} = -2\pi i \frac{1}{3!} = -\frac{\pi i}{3}$

4 Formula for Residues

The residue of a function $f(z)$ at its poles can be found using the below formulas:

<table>
<thead>
<tr>
<th>Residue at a simple pole</th>
</tr>
</thead>
<tbody>
<tr>
<td>We have 2 methods for evaluating the residue:</td>
</tr>
<tr>
<td>(i) $\text{Res}<em>{z=z_0} f(z) = \lim</em>{z \to z_0} (z - z_0) f(z)$</td>
</tr>
</tbody>
</table>
| (ii) If $f(z) = \frac{p(z)}{q(z)}$, $p(z_0) \neq 0$, $q(z_0) = 0$ and $q$ has a simple pole at $z_0$, then $f$ has a simple pole at $z_0$.  
  In this case, $\text{Res}_{z=z_0} f(z) = \frac{p(z_0)}{q'(z_0)}$ |

Example 14. Find the residue at $z = i$ of $f(z) = \frac{9z+i}{z^3+z}$.

Solution: Factorising the denominator, $f(z) = \frac{9z+i}{z^3+z} = \frac{9z+i}{z(z^2+1)}$.

$f(z)$ has a simple pole at $z = i$.

Using the first method,

$$\text{Res}_{z=i} \frac{9z+i}{z^3+z} = \lim_{z \to i} \frac{(z-i)(9z+i)}{z(z^2+1)(z-i)} = \lim_{z \to i} \frac{9z+i}{z(z^2+1)} = -5i$$

Using the second method,

$$\text{Res}_{z=i} \frac{9z+i}{z^3+z} = \frac{9z+i}{3z^2+1} \bigg|_{z=i} = -5i$$
Residue at a pole of order m
If \( f(z) \) has a pole of order \( m \) at \( z_0 \), the residue is \( \text{Res}_{z=z_0} f(z) = \frac{1}{(m-1)!} \lim_{z \to z_0} \left[ \frac{d^{m-1}}{dz^{m-1}} (z-z_0)^m f(z) \right] \)

Example 15. Find \( \text{Res}_{z=1} f(z) \), where \( f(z) = \frac{50z}{z^3 + z^2 - 4} \)
Solution:

Factorising the denominator, \( f(z) = \frac{50z}{z^3 + z^2 - 4} = \frac{50z}{(z+2)(z-1)^2} \).

\( f(z) \) has a pole of order 2 at \( z = 1 \).

\[
\text{Res}_{z=1} f(z) = \lim_{z \to 1} \left[ \frac{d}{dz} (z-1)^2 f(z) = \right]
\lim_{z \to 1} \left[ \frac{d}{dz} \left( \frac{50z}{z+4} \right) \right] = 8
\]

5 Cauchy Residue Theorem

Theorem 1. Let \( f(z) \) be analytic inside a simple closed path \( C \) and on \( C \), except for finitely many singular points \( z_1, z_2, \ldots, z_n \) inside \( C \).

Then the integral of \( f(z) \) taken anticlockwise around \( C \) equals \( 2\pi i \) times the sum of the residues of \( f(z) \) at \( z_1, z_2, \ldots, z_n \), i.e,

\[
\oint_C f(z) \, dz = 2\pi i \sum_{z=z_1} \text{Res}_{z=z_j} f(z)
\]

6 Applications of Cauchy Residue Theorem

6.1 Evaluation of definite integral using Residue theorem

Example 16. Evaluate the integral \( \oint_C \frac{4z}{z^2 - 3} \) anticlockwise around the given path

(i) \( |z| = 2 \)
(ii) \( |z| = 1/2 \)
(iii) \( |z-1| = 1/2 \)
(iv) \( |z-2| = 1/2 \)
Solution:

The function to be integrated has singularities at $z = 0, 1$.

The residues at these singularities are

$$Res_{z=0} \frac{4 - 3z}{z(z-1)} = -4, \quad Res_{z=1} \frac{4 - 3z}{z(z-1)} = 1$$

Now evaluate the integral over each path

(i) Both singularities are inside the contour, so by Residue theorem

$$\oint_{|z|=1/2} \frac{4 - 3z}{z^2 - z} = 2\pi i \left[ Res_{z=0} \frac{4 - 3z}{z^2 - z} + Res_{z=1} \frac{4 - 3z}{z^2 - z} \right] = -6\pi i$$

(ii) Only $z = 0$ is inside the contour, so again by Residue theorem

$$\oint_{|z|=1} \frac{4 - 3z}{z^2 - z} = 2\pi i \left[ Res_{z=0} \frac{4 - 3z}{z^2 - z} \right] = -8\pi i$$

(iii) Only $z = 1$ is inside the contour, so again by Residue theorem

$$\oint_{|z|=1} \frac{4 - 3z}{z^2 - z} = 2\pi i \left[ Res_{z=1} \frac{4 - 3z}{z^2 - z} \right] = 2\pi i$$

(iv) Neither of the singularities are inside the contour, so by Cauchy’s theorem

$$\oint_{|z|=1} \frac{4 - 3z}{z^2 - z} = 0$$

Example 17. Evaluate the integral $\oint_{|z|=3/2} \frac{\tan z}{z^2 - 1}$

Solution:

Here, we can rewrite the function to be integrated as $\frac{\tan z}{z^2 - 1} = \frac{\sin z}{(z+1)(z-1)\cos^2 z}$.

This function has singularities at $z = 1, -1, \frac{2n + 1}{2}$.

Of these, only $z = 1, -1$ are inside the contour.

Therefore, by Residue theorem

$$\oint_{|z|=3/2} \frac{\tan z}{z^2 - 1} = 2\pi i \left[ Res_{z=1} \frac{\tan z}{z^2 - 1} + Res_{z=-1} \frac{\tan z}{z^2 - 1} \right] = 2\pi \tan (1)$$

6.2 Integrals of rational functions of $\cos \theta$ and $\sin \theta$

We are interested in evaluating integrals of the form $I = \int_0^{2\pi} F(\cos \theta, \sin \theta) d\theta$.

To evaluate such integrals, we do the following:

(i) Let $z = e^{i\theta}$, so that $dz = ie^{i\theta} d\theta$, or $d\theta = \frac{dz}{ie^z}$. 

(ii) With this substitution, the definite integral on the interval \([0, 2\pi]\) becomes a contour integral over \(|z| = 1\).

(iii) From this substitution again, \(\cos(n\theta) = \frac{1}{2} \left( z^n + \frac{1}{z^n} \right)\), \(\sin(n\theta) = \frac{1}{2i} \left( z^n - \frac{1}{z^n} \right)\)

This will allow us to convert the given definite integral into a contour integral.

\[
I = \int_{0}^{2\pi} F(\cos \theta, \sin \theta) d\theta = \oint_{|z|=1} F \left( \frac{1}{2} \frac{z^n + 1}{z}, \frac{1}{2i} \frac{z^n - 1}{z} \right) \frac{dz}{iz}
\]

This new integral can be evaluated using Residue theorem.

Example 18. Evaluate \(\int_{0}^{2\pi} \frac{d\theta}{\sin \theta + 4 \cos \theta}\).

Solution:

Make the substitution \(\cos(\theta) = \left( \frac{1}{2} \frac{z^n + 1}{z} \right)\), \(d\theta = \frac{dz}{iz}\).

Then the integral can be converted to a contour integral

\[
\int_{0}^{2\pi} \frac{dz}{5 + 4 \cos \theta} = \oint_{|z|=1} \frac{1}{5 + 4 \left( \frac{1}{2} \frac{z^n + 1}{z} \right) iz} dz
\]

\[
= \frac{1}{i} \oint_{|z|=1} \frac{dz}{2z^2 + 5z + 2} = \frac{1}{i} \int_{|z|=1} \frac{dz}{(2z + 1)(z + 2)}
\]

Of the 2 singularities \(z = -2, -1/2\), the simple pole \(z = -1/2\) lies inside \(|z| = 1\).

\[
\text{Res}_{z=-1/2} \frac{1}{(2z + 1)(z + 2)} = \frac{1}{4z + 5} \bigg|_{z=-1/2} = \frac{1}{3}
\]

\[
\int_{0}^{2\pi} \frac{dz}{5 + 4 \cos \theta} = \frac{1}{i} \oint_{|z|=1} \frac{dz}{2z^2 + 5z + 2} = \frac{1}{i} \frac{2\pi}{3} = \frac{2\pi}{3}
\]

Example 19. Evaluate \(\int_{0}^{2\pi} \frac{\cos(2\theta)}{5 + 4 \cos \theta} d\theta\).

Solution:

Make the substitution \(\cos(\theta) = \left( \frac{1}{2} \frac{z^n + 1}{z} \right)\), \(\cos(2\theta) = \left( \frac{1}{2} \frac{z^n + 1}{z} \right)^2\), \(d\theta = \frac{dz}{iz}\).

Then the integral can be converted to a contour integral

\[
\int_{0}^{2\pi} \frac{\cos(2\theta)}{5 + 4 \cos \theta} d\theta = \oint_{|z|=1} \frac{1}{2i} \frac{z^n + 1}{z^2(2z^2 + 5z + 2)}
\]

The function \(f(z) = \frac{z^n + 1}{z^2(2z^2 + 5z + 2)}\) has singularities \(z = 0, 0, -2, -1/2\).

Of these, \(z = 0\) and \(z = -1/2\) lie inside \(|z| = 1\). \(z = 0\) is a pole of order 2 and \(z = -1/2\) a simple pole.
\[ \text{Res}_{z=-1} f(z) = \left. \frac{z^4 + 1}{8z^3 + 15z^2 + 4z} \right|_{z=-1/2} = \frac{17}{12} \]

\[ \text{Res}_{z=0} f(z) = \frac{1}{(2-1)!} \lim_{z \to 0} \frac{d}{dz} \left[ \frac{z^2}{z^2(2z^2 + 5z + 2)} \right] = \frac{5}{4} \]

\[ \int_0^{2\pi} \frac{\cos(2\theta)}{5+4\cos\theta} d\theta = \int_{|z|=1} \frac{1}{2i} \frac{z^4 + 1}{z^2(2z^2 + 5z + 2)} = \frac{1}{2i} \frac{17}{12} - \frac{5}{4} = \pi \]

6.3 Improper integrals of the form \( \int_{-\infty}^{\infty} \frac{f(x)}{g(x)} dx \) with \( \text{deg}(g(x)) - \text{deg}(f(x)) \geq 2 \)

**Example 20.** Evaluate \( \int_{-\infty}^{\infty} \frac{x^2}{(x^2+1)^2} dx \).

**Solution:**

\[ f(x) = \frac{x^2}{(x^2+1)^2}, \]

The singular points of the above function are given by \( z = \pm i, \pm i \).

\( z = \pm i \) are poles of order 2.

Of these, only \( z = i \) lies in the upper half plane.

\[ \int_{-\infty}^{\infty} \frac{x^2}{(x^2+1)^2} dx = 2\pi i \text{Res}_{z=i} \left( \frac{f(z)}{g(z)} \right) \]

\[ \text{Res}_{z=i} \frac{f(z)}{g(z)} = \frac{1}{(2-1)!} \lim_{z \to i} \frac{d}{dz} \left[ (z-i)^2 \frac{z^2}{(z^2+1)^2} \right] = \frac{1}{4i} \]

Therefore

\[ \int_{-\infty}^{\infty} \frac{x^2}{(x^2+1)^2} dx = 2\pi i \frac{1}{4i} = \frac{\pi}{2} \]
Example 21. Evaluate \( \int_0^\infty \frac{1}{(x^2+a^2)^2} \, dx \).

Solution:

The function to be integrated is even, so we can apply the following result:

\[
\int_{-\infty}^{\infty} f(x) \, dx = 2 \int_0^\infty f(x) \, dx
\]

Then \( \int_0^\infty \frac{1}{(x^2+a^2)^2} \, dx = \frac{1}{2} \int_{-\infty}^{\infty} \frac{1}{(x^2+a^2)^2} \, dx \)

\[
\frac{1}{g(z)} = \frac{1}{(z^2+a^2)^2}
\]

The singular points of the above function are given by \( z = \pm ai, \pm ai \).

\( z = \pm ai \) are poles of order 2.

Of these, only \( z = ai \) lies in the upper half plane.

Therefore, the integral is

\[
\frac{1}{2} \int_{-\infty}^{\infty} \frac{1}{(x^2+a^2)^2} \, dx = \frac{1}{2} \frac{2\pi i \text{Res}_{z=ai} f(z)}{g(z)}
\]

\[
\frac{\pi}{2} \frac{1}{(2-1)!} \lim_{x \to ai} \frac{d}{dz} (z-ai)^2 \frac{1}{(z^2+a^2)^2} = \frac{\pi}{4a^3}
\]

6.4 Integrals of the form \( \int_{-\infty}^{\infty} \frac{f(z)}{g(z)} \cos(mx) \, dx \) and \( \int_{-\infty}^{\infty} \frac{f(z)}{g(z)} \sin(mx) \, dx \)

Method of evaluation

\[
\int_{-\infty}^{\infty} \frac{f(x)}{g(x)} \cos(mx) \, dx = Re \left( \oint_C \frac{f(z)}{g(z)} e^{imz} \, dz \right)
\]

\[
\int_{-\infty}^{\infty} \frac{f(x)}{g(x)} \sin(mx) \, dx = Im \left( \oint_C \frac{f(z)}{g(z)} e^{imz} \, dz \right)
\]

where \( C \) is a contour enclosing singularities of \( \frac{f(z)}{g(z)} \) in the upper half plane.

Example 22. Show that \( \int_0^\infty \frac{\cos ax}{1-x^2} \, dx = \frac{\pi}{2} e^{-a} \).

Solution:
\[ \frac{I(x)}{g(x)} e^{inx} = \frac{e^{-ax}}{1 + x^2}. \]

The singular points of the above function are given by \( z = \pm i \).

Of these, only \( z = i \) is in the upper half plane.

So

\[ \int_0^\infty \frac{\cos ax}{1 + x^2} \, dx = \left. \frac{1}{2} \int_{-\infty}^\infty \frac{\cos ax}{1 + x^2} \, dx \right|_{C} = \frac{1}{2} \text{Re} \left( \int_C \frac{e^{iaz}}{(1 + z^2)} \, dz \right) \]

where \( C \) is a contour enclosing the pole \( z = i \) in the upper half plane.

Then the value of this integral is

\[ = \frac{1}{2} \text{Re} \left( 2\pi i \text{Res}_{z=i} \frac{e^{iaz}}{(1 + z^2)} \right) = \frac{1}{2} \text{Re} \left( \frac{2\pi i e^{-a}}{2i} \right) = \frac{\pi e^{-a}}{2} \]

**Example 23.** Evaluate \( \int_{-\infty}^{\infty} \frac{z \sin x}{(x^2+a^2)} \, dx, \quad a > 0 \)

**Solution:**

\[ \frac{I(x)}{g(x)} e^{inx} = \frac{e^{-ax}}{(x^2+a^2)}. \]

The singular points of the above function are \( z = \pm ai \).

Of these, only \( z = ai \) is in the upper half plane.

So

\[ \int_{-\infty}^{\infty} \frac{z \sin x}{(x^2+a^2)} \, dx = \text{Im} \left( \int_C \frac{ze^{iz}}{(z^2 + a^2)} \, dz \right) \]

where \( C \) is a contour enclosing the pole \( z = ai \) in the upper half plane.

The value of this integral is

\[ = \text{Im} \left( 2\pi i \text{Res}_{z=ai} \frac{ze^{iz}}{(z^2 + a^2)} \right) = \text{Im} \left( 2\pi i \frac{e^{-a}}{2} \right) = \text{Im}(\pi i e^{-a}) = \pi e^{-a} \]

**References**


ECT201 SOLID STATE DEVICES
COURSE INFORMATION SHEET

PROGRAMME: ELECTRONICS AND COMMUNICATION ENGINEERING

COURSE: SOLID STATE DEVICES

COURSE CODE: ECT 201
REGULATION: 2019
COURSE AREA/DOMAIN: ELECTRONICS
CORRESPONDING LAB COURSE CODE (IF ANY): NIL

DEGREE: B.TECH

SEMESTER: 3
CREDITS: 4
COURSE TYPE: CORE
CONTACT HOURS: 3+1 (Tutorial) Hours/Week.
LAB COURSE NAME: NA

SYLLABUS:

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<td>I</td>
<td>Elemental and compound semiconductors, Intrinsic and Extrinsic semiconductors, concept of effective mass, Fermions-Fermi Dirac distribution, Fermi level, Doping &amp; Energy band diagram, Equilibrium and steady state conditions, Density of states &amp; Effective density of states, Equilibrium concentration of electrons and holes. Excess carriers in semiconductors: Generation and recombination mechanisms of excess carriers, quasi Fermi levels.</td>
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<td>II</td>
<td>Carrier transport in semiconductors, drift, conductivity and mobility, variation of mobility with temperature and doping, Hall Effect. Diffusion, Einstein relations, Poisson equations, Continuity equations, Current flow equations, Diffusion length, Gradient of quasi Fermi level</td>
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<td>III</td>
<td>PN junctions : Contact potential, Electrical Field, Potential and Charge distribution at the junction, Biasing and Energy band diagrams, Ideal diode equation. Metal Semiconductor contacts, Electron affinity and work function, Ohmic and Rectifying Contacts, current voltage characteristics. Bipolar junction transistor, current components, Transistor action, Base width modulation.</td>
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<td>IV</td>
<td>Ideal MOS capacitor, band diagrams at equilibrium, accumulation, depletion and inversion, threshold voltage, body effect, MOSFET-structure, types, Drain current equation (derive)-linear and saturation region, Drain characteristics, transfer characteristics.</td>
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<tr>
<td>V</td>
<td>MOSFET scaling – need for scaling, constant voltage scaling and constant field scaling. Sub threshold conduction in MOS.</td>
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Short channel effects- Channel length modulation, Drain Induced Barrier Lowering, Velocity Saturation, Threshold Voltage Variations and Hot Carrier Effects. Non-Planar MOSFETs: Fin FET – Structure, operation and advantages

TEXT/REFERENCE BOOKS:

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<tr>
<td>T1</td>
<td>Ben G. Streetman and Sanjay Kumar Banerjee, Solid State Electronic Devices, Pearson 6/e, 2010 (Modules I, II and III)</td>
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<td>R1</td>
<td>Neamen, Semiconductor Physics and Devices, McGraw Hill, 4/e, 2012</td>
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<td>Sze S.M., Semiconductor Devices: Physics and Technology, John Wiley, 3/e, 2005</td>
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<td>R3</td>
<td>Pierret, Semiconductor Devices Fundamentals, Pearson, 2006</td>
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<td>R4</td>
<td>Sze S.M., Physics of Semiconductor Devices, John Wiley, 3/e, 2005</td>
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<td>R6</td>
<td>Yannis Tsividis, Operation and Modelling of the MOS Transistor, Oxford University Press.</td>
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<td>R7</td>
<td>Jan M.Rabaey, Anantha Chandrakasan, Borivoje Nikolic, Digital Integrated Circuits – A Design Perspective, PHI.</td>
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COURSE PRE-REQUISITES:

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<td>To develop basic idea about calculus and differential equations.</td>
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<tr>
<td></td>
<td>ENGINEERING PHYSICS</td>
<td>To have a basic idea of semiconductor devices, Quantum mechanics, LEDs, laser diodes etc.</td>
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COURSE OBJECTIVES:

1. To provide an insight into the basic semiconductor concepts.
2. To provide a sound understanding of current semiconductor devices and technology to appreciate its applications to electronics circuits and system

COURSE OUTCOMES:

<table>
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<th>Sl. No.</th>
<th>DESCRIPTION</th>
<th>Blooms’ Taxonomy Level</th>
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</table>
1. Graduates will be able to **define** and **understand** the concepts in semiconductor physics.

   Knowledge & Understand (level 1,2)

2. Graduates will be able to **describe** and **apply** the generation and recombination processes in semiconductors.

   Understand & Apply (level 2,3)

3. Graduates will be able to **explain** the structure, creation of electric field and working of PN junction semiconductor diodes.

   Understand (level 2)

4. Graduates will be able to **illustrate** the minority carrier distribution across PN junction semiconductor diodes.

   Apply (level 3)

5. Graduates will **develop** skills and can do research in new concepts and devices.

   Create (level 6)

6. Graduates can **summarize** concepts that studied relating different modes of operation and the various current components in BJTs and **analyze** energy band diagram of PN junction diodes, BJTs, metal semiconductor junctions and MOS capacitors.

   Evaluate & Analyze (level 5,4)

### CO-PO AND CO-PSO MAPPING

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### JUSTIFICATIONS FOR CO-PO-PSO MAPPING

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<td>Application of fundamental knowledge in semiconductor physics</td>
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<td>PO1-CO.2,3,4,5,6</td>
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<td>Application of basic knowledge to complex engineering problems</td>
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</table>
PO2-CO.1,2,3,4,5,6  2  Identification and analysis of complex engineering problems

PO4-CO.2,3,4,6  1  Analysis and produce valid conclusions

PO4-CO.5  3  develop new skills and to produce valid conclusions

PO12-CO1  3  understanding of semiconductor physics will help them in life long learning

PO12-CO5  2  Research in new concepts helps them in independent learning

PSO1-CO.1  2  Knowledge in semiconductor physics will help them in VLSI systems

PSO2-CO.1  3  Knowledge in semiconductor physics will help them in developing applications using EDA tools

GAPS IN THE SYLLABUS - TO MEET INDUSTRY/PROFESSION REQUIREMENTS:

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<td>Fabrication of PN junctions, FETs etc.</td>
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<td>2</td>
<td>Physics of HEMT devices</td>
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PROPOSED ACTIONS: TOPICS BEYOND SYLLABUS/ASSIGNMENT/INDUSTRY VISIT/GUEST LECTURER/NPTEL ETC

TOPICS BEYOND SYLLABUS/ADVANCED TOPICS/DESIGN:

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<td>SPICE models</td>
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WEB SOURCE REFERENCES:

2  http://nptel.iitm.ac.in/video.php?subjectId=117106091
3  http://nptel.iitm.ac.in/courses/Webcourse-contents/IIT-Delhi/Semiconductor%20Devices/index.htm
4  http://nptel.iitm.ac.in/courses/Webcourse-contents/IIT-%20Guwahati/ic_tech/index.html
5  http://education.jlab.org/itselemental/ele014.html
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DELIBERY/INSTRUCTIONAL METHODOLOGIES:

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ASSESSMENT METHODOLOGIES-DIRECT

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<th>✓ TESTS/MODEL EXAMS</th>
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<td>☐ MINI/MAJOR PROJECTS</td>
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ASSESSMENT METHODOLOGIES-INDIRECT

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Prepared by
Bonifus P.L
(Faculty)

Approved by
Dr. Rithu James
(HOD)
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ASSIGNMENT – 1

Answer All Question

1. Derive the expression for drain current at saturation for a MOSFET. (6 marks)

2. With the help of necessary band diagrams, explain equilibrium, accumulation, depletion and inversion stages of a MOS capacitor. (6 marks)
3. Explain the drain characteristics and transfer characteristics of an enhancement type MOSFET with suitable diagrams. (6 marks)

Multiple Choice Questions (1 mark each)

4. Which of the following terminals do not belong to a MOSFET?
   a) Drain
   b) Gate
   c) Base
   d) Source

5. MOSFET is
   a) An uncontrolled device
   b) A voltage controlled device
   c) A current controlled device
   d) A temperature controlled device

6. In the transfer characteristics of a MOSFET, the threshold voltage is the measure of the
   a) Minimum voltage to induce an n-channel/p-channel for conduction
   b) Minimum voltage till which drain current is constant
   c) Minimum voltage to turn off the device
   d) None of the above

7. Consider an ideal enhancement type MOSFET. If \( V_{GS} = 0 \) V, then \( I_D = ? \)
   a) Zero
   b) Maximum
   c) \( I_{DSAT} \)
   d) None of the above

8. The region of operation of an enhancement type MOSFET when \( V_{GS} > V_{TO} \) and \( V_{DS} < V_{DSAT} \) is
   a) Linear
   b) Saturation
   c) Cut-Off
   d) Pinch Off

9. A MOS transistor with no conducting channel at zero bias is
   a) Depletion Type
   b) Enhancement Type
   c) n – channel
   d) p – channel

10. In the saturation region of operation of a MOSFET
    a) \( V_{GS} = V_{DS} \)
b) \( V_{DSAT} = V_{GS} \)

c) \( V_{DS} > V_{GS} - V_{TO} \)

d) \( V_{DS} < V_{GS} - V_{TO} \)

**ASSIGNMENT - 2**

1) Calculate the thermal equilibrium electron and hole concentration in silicon at \( T=300K \), when the Fermi energy level is 0.27 eV below the conduction band edge \( E_C \). The effective densities of states in the conduction band and valence band are 2.8 x 10^{19} \text{cm}^{-3} and 1.04 x 10^{19} \text{cm}^{-3} respectively at 300K.

2) A Ge sample is doped with 10^{16} Boron atoms/cc. Find the electron concentration. Intrinsic carrier concentration of Ge is 2.5 x 10^{13} \text{cm}^{-3} at 300K.

3) For the given data, calculate hole and intrinsic carrier concentrations. Also sketch the band diagram. \( N_C = 10^{19} \text{cm}^{-3}, N_V = 5 \times 10^{16} \text{cm}^{-3}, E_g=2\text{eV}, T=900K, n_0=10^{17} \text{cm}^{-3} \).

4) A Silicon sample is doped with 10^{17} boron atoms/cm^3. What is the equilibrium electron and hole concentrations at 300K? Where is \( E_F \) relative to \( E_i \)? Draw the energy band diagram. Intrinsic carrier concentration of Silicon is 1.5 x 10^{10} at 300K.

5) A Ge sample is doped with 10^{17} Boron atoms/cc. Determine the carrier concentrations and Fermi level position at room temperature. \( n_i \) for Ge is 2.5 x 10^{13}/cc at room temperature.

6) For a Si sample at 300K, \( p_0 = 4 \times 10^{12}/\text{cc} \). Determine a) the electron density b) the acceptor density if donor density is 10^{12}/cc.

7) A semiconductor is known to have a bandgap of 1.25 eV and intrinsic carrier concentration of 1.6 x 10^{10}/cc at room temperature. Estimate \( N_C \) and \( N_V \), if \( m_n^* : m_p^* = 4:1 \). Assume \( kT = 0.025 \text{ eV} \).

**TUTORIALS**

**MODULE – I:**

1) An unknown semiconductor has \( E_g = 1.1 \text{ eV} \) and \( N_C = N_V \). It is doped with 10^{15} \text{cm}^{-3} donors, where the donor level is 0.2 eV below \( E_C \). Given that \( E_g = 0.25\text{eV} \) below \( E_C \), calculate \( n_i \) and the concentration of electrons and holes in the semiconductor at 300 K.

2) A Si sample is doped with 10^{17} As atoms/cm^3. What is the equilibrium \( h^+ \) concentration \( p_0 \) at 300 K? Where is \( E_F \) relative to \( E_i \)?

3) A Si sample is doped with 10^{16} \text{cm}^{-3} boron atoms and a certain number of shallow donors. The Fermi level is 0.36 eV above \( E_i \) at 300 K. What is the donor concentration \( N_D \)?

4) If \( n_0 = 10^{16} \text{cm}^{-3} \), where is the Fermi level relative to \( E_i \) in Si at 300 K?

5) A new semiconductor has \( N_C = 10^{19} \text{cm}^{-3} \), \( N_V = 5 \times 10^{18} \text{cm}^{-3} \), and \( E_g = 2 \text{ eV} \). If it is doped with 10^{17} donors (fully ionized), calculate the electron, hole, and intrinsic carrier concentrations at 627°C. Sketch the simplified band diagram, showing the position of \( E_F \)

6) A semiconductor device requires n-type material; it is to be operated at 400K. Would Si doped with 10^{15} \text{atoms/cm}^3 \) of arsenic be useful in this application? Could Ge doped with 10^{15} \text{cm}^{-3} antimony be used?

7) A silicon sample is doped with 5 x 10^{16} Arsenic atoms/cc and 3 x 10^{16} Boron atoms/cc. Determine

i. \( \bar{\varepsilon} \) and \( h^+ \) concentrations at room temperature; &

ii. Position of Fermi level (Assume \( n_i = 1.5 \times 10^{10} \text{cc at room temperature} \)

8) A Ge sample is doped with 5 x 10^{13} As atoms/cm^3. Determine the carrier concentration and Fermi level position at 300K.
9) A Si sample is doped with As atoms with concentration $10^{16}$ atoms/cc. It is also steadily illuminated such that $g_{op} = 10^{21}$ EHP/cc. It $\tau_n = \tau_p = 10^{-6}$ s, calculate separation between quasi Fermi levels and show the positions of equilibrium and quasi Fermi levels at 300K.

10) Assume that 1013 EHP/cm3 are created optically every microsecond in a Si sample with $n_0 = 1014$ cm-3 and $\tau_n = \tau_p = 2\mu$sec.
   a) Find steady state excess $\bar{\epsilon}(or \ h^+)$ concentration
   b) Find minority carrier concentration at equilibrium
   c) Find carrier concentration at steady state
   d) Find quasi-Fermi levels for $\bar{\epsilon}$s & $h^+$s

11) A Si sample with $10^{16}$/cm$^3$ donors is optically excited such that $10^{19}$/cm$^3$ electron-hole pairs are generated per second uniformly in the sample. The laser causes the sample to heat up to 450 K. Find the quasi-Fermi levels and the change in conductivity of the sample upon shining the light. Electron and hole lifetimes are both 10 $\mu$s. $D_p = 12$ cm$^2$/s; $D_n = 36$ cm$^2$/s; $n_i = 10^{14}$ cm$^{-3}$ at 450 K. What is the change in conductivity upon shining light?

12) A Si sample with $10^{15}$/cm$^3$ donors is uniformly optically excited at room temperature such that $10^{19}$/cm$^3$ electron-hole pairs are generated per second. Find the separation of the quasi-Fermi levels and the change of conductivity upon shining the light. Electron and hole lifetimes are both 10 $\mu$s. $D_p = 12$ cm$^2$/s, $\mu_n = 1300$ cm$^2$/V-s.

13) An n-type Si sample with $N_d = 10^{15}$ cm$^{-3}$ is steadily illuminated such that $g_{op} = 10^{21}$ EHP/cm$^2$s. If $\tau_n = \tau_p = 10^{-6}$ s, calculate the separation in the quasi-Fermi levels, $(F_n - F_p)$. Draw band diagram.

14) Derive an expression relating the intrinsic level $E_i$ to the center of the band gap $E_g/2$. Calculate the displacement of $E_i$ from $E_g/2$ for Si at 300 K, assuming the effective mass values for electrons and holes are 1.1$m_0$ and 0.56$m_0$ respectively.

**MODULE - II**

1) A Si bar 1 $\mu$m long and 100$\mu$m$^2$ in cross-sectional area is doped with $10^{17}$ cm$^{-3}$ Ph. a) Find the current at 300K with 10V applied. b) How long does it take an average electron to drift 1 $\mu$m in pure Si at an electric field of 100V/cm? Repeat for 10$^5$V/cm.

2) A 2cm long piece of Si with a cross-sectional area of 0.1cm$^2$ is doped with donors at $10^{15}$/cm$^3$ and has a resistance of 90 ohms. The saturation velocity of electrons in Si is $10^7$ cm/s for fields above $10^5$ V/cm. Calculate the electron drift velocity if we apply a voltage of 100V across the piece. What is the current through the piece if we apply a voltage of $10^6$ V across it?

3) a) Show that the minimum conductivity of a semiconductor sample occurs when $n_0 = n_i^*(\mu_p/ \mu_n)^{1/2}$.
   b) What is the expression for the minimum conductivity $\sigma_{min}$?
   c) Calculate $\sigma_{min}$ for Si at 300 K and compare with the intrinsic conductivity.

4) A Si sample is doped with $10^{17}$ boron atoms/cm$^3$. What is the electron concentration $n_0$ at 300 K? What is the resistivity? $\mu_n = 1350$ cm$^2$/Vs and $\mu_p = 250$ cm$^2$/Vs.

5) For a Si conductor of length 5 $\mu$m, doped n-type at $10^{15}$ cm$^{-3}$, calculate the current density for an applied voltage of 2.5 V across its length. How about for a voltage of 2500 V? The electron and hole mobilities are 1500 cm$^2$/V-s and 500 cm$^2$/V-s, respectively, in the ohmic region for electric fields below $10^4$ V/cm. For higher fields, electrons and holes have a saturation velocity of $10^7$ cm/s.

6) A Silicon bar of 100 cm long and 1 cm$^2$ cross sectional area is doped with $10^{17}$Arsenic atoms/cm$^3$. Calculate electron and hole concentrations at 300K. Also find the conductivity and the current with 10V applied. Electron mobility at this doping is 700 cm$^2$/V-sec.
7) Consider a semiconductor bar with $w=0.1\text{mm}$, $t=10\mu\text{m}$ and $L=5\text{mm}$. For $B=10\text{kg}$ ($1\text{kg}=10^{-5}\text{Wb/cm}^2$) and a current $1\text{mA}$, we have $V_{AB}=-2\text{mV}$, $V_{CD}=100\text{mV}$. Find the type, concentration, and mobility of the majority carrier.

8) An intrinsic Si sample is doped with donors from one side such that $N_d = N_0 e^{-ax}$.
   a) Find an expression for the built-in electric field at equilibrium over the range for which $N_d \gg n_i$.
   b) Evaluate the field when $a = 1(\mu\text{m})^{-1}$.
   c) Sketch a band diagram and indicate the direction of the field.

9) A Si sample with $10^{16}/\text{cm}^3$ donors is optically excited such that $10^{19}/\text{cm}^3$ electron-hole pairs are generated per second uniformly in the sample. The laser causes the sample to heat up to $450\text{ K}$. Find the change in conductivity of the sample upon shining the light. Electron and hole lifetimes are both $10\ \mu\text{s}$. $D_p = 12\ \text{cm}^2/\text{s}$; $D_n = 36\ \text{cm}^2/\text{s}$; $n_i = 10^{14}\ \text{cm}^{-3}$ at $450\text{ K}$.

10) A Si sample with $10^{15}/\text{cm}^3$ donors is uniformly optically excited at room temperature such that $10^{19}/\text{cm}^3$ electron-hole pairs are generated per second. Find the change of conductivity upon shining the light. Electron and hole lifetimes are both $10\ \mu\text{s}$. $D_p = 12\ \text{cm}^2/\text{s}$, $\mu_n = 1300\ \text{cm}^2/\text{V-s}$.

11) In a very long p-type Si bar with cross-sectional area = $0.5\ \text{cm}^2$ and $N_a = 10^{17}\ \text{cm}^{-3}$, we inject holes such that the steady state excess hole concentration is $5 \times 10^{16}\ \text{cm}^{-3}$ at $x = 0$. What is the steady state separation between $F_p$ and $E_i$ at $x = 1000\ \text{A}$? What is the hole current there? How much is the excess stored hole charge? Assume that $u_p = 500\ \text{cm}^2/\text{V-s}$ and $\tau_p = 10^{-10}\text{s}$.

**MODULE – III**

1) Extend the equation \( I_E \approx qA (D_p/L_p) \Delta p E \coth(Wh/L_p) \) to include the effects of non-unity emitter injection efficiency ($\gamma < 1$). Derive an equation for $\gamma$. Assume that the emitter region is long compared with an electron diffusion length.

2) Assume that a p-n-p transistor is doped such that the emitter doping is 10 times that in the base, the minority carrier mobility in the emitter is one-half that in the base, and the base width is one-tenth the minority carrier diffusion length. The carrier lifetimes are equal. Calculate $\alpha$, $\beta$ and $\gamma$ for this transistor.

3) For a p-n-p BJT with $NE > NB > NC$, show the dominant current components, with proper arrows, for directions in the normal active mode. If $I_E = 10\ \text{mA}$, $I_E = 100\ \mu\text{A}$, $I_C = 9.8\ \text{mA}$, and $I_C = 1\ \mu\text{A}$, calculate the base transport factor, emitter injection efficiency, common-base current gain, common-emitter current gain, and $ICBO$ if the minority stored base charge is $4.9 \times 10^{-11}\ \text{C}$, calculate the base transit time.

4) An abrupt Si p-n junction has $Na = 1018\ \text{cm}^{-3}$ on one side and $Nd = 5 \times 1015\ \text{cm}^{-3}$ on the other. Calculate the Fermi level positions at 300 $\text{K}$ in the p and n regions. Draw an equilibrium band diagram for the junction and determine the contact potential $V_0$ from the diagram. Compare the results of part (b) with $V_0$ as calculated from equation.

5) An abrupt Si p-n junction has $Na = 1018\ \text{cm}^{-3}$ on one side and $Nd = 5 \times 1015\ \text{cm}^{-3}$ on the other. The junction has a circular cross section with a diameter of $10\mu\text{m}$. Calculate $x_n0$, $x_p0$, $Q+$ and $\xi0$ for this junction at equilibrium(300 $\text{K}$).

6) Boron is implanted into an n-type Si sample (Nd=1016cm-3) forming an abrupt junction of square cross-section, with area $2\times10^{-3}\text{cm}^2$. Assume that acceptor concentration in p-type region is $Na= 4\times1018\ \text{cm}^{-3}$.

   Calculate $V_0$, $x_n0$, $x_p0$, $Q+$ and $\xi0$ for this junction at equilibrium(300K). Sketch $\xi(x)$ and charge density.

7) In a p-n+ junction, the n side has a donor concentration of $1016\ \text{cm}^{-3}$. If $n_i = 1010\text{cm}^{-3}$, the relative dielectric constant $\varepsilon_r = 12$, $D_n = 50\ \text{cm}^2/s$, $D_p = 20\ \text{cm}^2/s$, and the electron and
hole minority carriers have lifetimes $\tau = 100$ ns and 50 ns, respectively, and a forward bias of 0.6 V, calculate the hole diffusion current density $2 \mu m$ from the depletion edge on the n side. If we double the p+ doping, what effect will it have on this hole diffusion current?  
8) Assume that the doping concentration $Na$ on the p side of an abrupt junction is the same as N$D$ on the n side. Each side is many diffusion lengths long. Find the expression for the hole current $I_p$ in the p-type material.
9) Show that the peak electric field in the transition region is controlled by the doping on the more lightly doped side of the junction.
10) A Si p-n junction with cross-sectional area $A = 0.001$ cm$^2$ is formed with $Na = 1015$ cm$^{-3}$, $N_D = 1017$ cm$^{-3}$. Calculate:
   (a) Contact potential, $V_0$.
   (b) Space-charge width at equilibrium (zero bias).
   (c) Current with a forward bias of 0.5 V. Assume that the current is diffusion dominated. Assume $\mu_n = 1500$ cm$^2$/V-s, $\mu_p = 450$ cm$^2$/V-s, $\tau_n = \tau_p = 2.5$ ms. Which carries most of the current, electrons or holes, and why? If you wanted to double the electron current, what should you do?

**MODULE – IV**

1) Calculate the threshold voltage $V_{TO}$ at $V_{SB} = 0$, for a polysilicon gate n-channel MOS transistor, with the following parameters: substrate doping density $N_A = 10^{16}$ cm$^{-3}$, polysilicon gate doping density $N_D = 2 \times 10^{20}$ cm$^{-3}$, gate oxide thickness $t_{ox} = 500$ A, and oxide-interface fixed charge density $Nox = 4 \times 10^{10}$ cm$^{-2}$.
2) A pMOS transistor was fabricated on an n-type substrate with a bulk doping density of $N_D = 10^{16}$ cm$^{-3}$, gate doping density (n-type poly) of $N_{Na} = 1020$ cm$^{-3}$, $Q_{ox}/q = 4 \times 10^{10}$ cm$^{-2}$, and gate oxide thickness of $t_{ox} = 0.1$ $\mu$m. Calculate the threshold voltage at room temperature for $V_{SB} = 0$. Use $\varepsilon_{Si} = 11.7 \varepsilon_o$.
3) Consider a MOS system with the following parameters:
   - $t_{ox} = 200$ A
   - $\phi_{GC} = -0.85$ V
   - $N_A = 2 \times 10^{15}$ cm$^{-3}$
   - $Q_{ox} = q \times 2 \times 10^{11} C/cm^2$
   - (a) Determine the threshold voltage $V_{TO}$ under zero bias at room temperature ($T = 300$ K). Note that $\varepsilon_{ox} = 3.97 \varepsilon_o$ and $\varepsilon_{Si} = 11.7 \varepsilon_o$.
   - (b) Determine the type (p-type or n-type) and amount of channel implant ($N_i/cm^2$) required to change the threshold voltage to 0.8 V.
4) For a MOSFET with gate oxide thickness = 100 A, $VT = 1$ V and $W = 50 \mu$m, $L = 2 \mu$m, calculate the drain current at $VGS = 5$ V, $VDS = 0.1$ V. Repeat for $VGS = 3$ V, $VDS = 5$ V. Assume an electron channel mobility $\mu_n = 200$ cm$^2$/V-s, and the substrate is connected to the source.  
5) Calculate the VT of a Si n-channel MOSFET for a gate-to-substrate work function difference $\phi_{GC} = -1.5$ V, gate oxide thickness = 100 A, $NA = 1018$ cm$^{-3}$, and fixed oxide charge of $5 \times 10^{10}$ q C/cm$^2$, for a substrate bias of -2.5 V. At VT, what are the electron and hole concentrations at the oxide-Si interface and deep in the substrate?  
6) Find the maximum depletion width, minimum capacitance $C_i$, and threshold voltage for an ideal MOS capacitor with a 10 nm gate oxide ($SiO_2$) on p-type Si with $Na = 1016$ cm$^{-3}$. Next, include the effects of flat band voltage, assuming an n+ polysilicon gate and fixed oxide charge of $5 \times 10^{10}$ q C/cm$^2$.
7) Calculate the VT of a Si-P-channel MOS transistor for an n+-polysilicon gate with silicon oxide thickness = 50 A, $Nd = 1 \times 1018$ cm$^{-3}$ and a fixed charge of $2 \times 10^{10}q$ C/cm$^2$. What B dose is required to change the VT to 0 V? Assume a shallow B implant. $\phi_{GC} = -0.1$ V.
8) A Si n-channel MOSFET has $\mu n = 600 \text{ cm}^2/\text{V sec}$, $C_{ox} = 1.2 \times 10^{-7} \text{ F/cm}^2$, $W = 50 \mu\text{m}$, $L = 10 \mu\text{m}$ and $V_{TH} = 0.8\text{V}$. Find the drain current when
   - $V_{GS} = 2\text{V}$ and $V_{DS} = 1\text{V}$
   - $V_{GS} = 3\text{V}$ and $V_{DS} = 5\text{V}$

9) For a long channel n-MOSFET with $V_T = 1\text{V}$, calculate the $V_{GS}$ required for an $I_D(\text{sat.})$ of 0.1 mA and $V_D(\text{sat.})$ of 5 V. Calculate the small-signal output conductance $g$ and the transconductance $g_m(\text{sat.})$ at $V_{DS} = 10\text{V}$. Recalculate the new $I_D$ for $V_{GS} - V_T = 3\text{V}$ and $V_{DS} = 4\text{V}$. 
ECT 203 LOGIC CIRCUIT DESIGN
COURSE INFORMATION SHEET

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SYLLABUS:

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<td>I</td>
<td>Binary and hexadecimal number systems; Methods of base conversions; Binary and hexadecimal arithmetic; Representation of signed numbers; Fixed and floating point numbers; Binary coded decimal codes; Gray codes; Excess 3 code. Alphanumeric codes: ASCII. Basics of verilog -- basic language elements: identifiers, data objects, scalar data types, operators.</td>
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<td>II</td>
<td>Boolean postulates and laws – Logic Functions and Gates De-Morgan’s Theorems, Principle of Duality, Minimization of Boolean expressions, Sum of Products (SOP), Product of Sums (POS), Canonical forms, Karnaugh map Minimization. Modeling in verilog, Implementation of gates with simple verilog codes.</td>
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<td>III</td>
<td>Combinatorial Logic Systems - Comparators, Multiplexers, Demultiplexers, Encoder, Decoder. Half and Full Adders, Subtractors, Serial and Parallel Adders, BCD Adder. Modeling and simulation of combinational circuits with verilog codes at the gate level</td>
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<td>IV</td>
<td>Building blocks like S-R, JK and Master-Slave JK FF, Edge triggered FF, Conversion of Flipflops, Excitation table and characteristic equation. Implementation with verilog codes. Ripple and Synchronous counters and implementation in verilog, Shift registers-SIPO, SISO, PISO, PIPO. Shift Registers with parallel Load/Shift, Ring counter and Johnsons counter. Asynchronous and Synchronous counter design, Mod N counter. Modeling and simulation of flipflops and counters in verilog.</td>
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<td>V</td>
<td>TTL, ECL, CMOS - Electrical characteristics of logic gates – logic levels and noise margins, fan-out, propagation delay, transition time, power consumption and power-delay product. TTL inverter - circuit description and operation; CMOS inverter - circuit description and operation; Structure and operations of TTL and CMOS gates; NAND in TTL and CMOS, NAND and NOR in CMOS.</td>
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**TOTAL HOURS**: 55

**TEXT/REFERENCE BOOKS:**

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COURSE PRE-REQUISITES:

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COURSE OBJECTIVES:

1. Impart the basic knowledge of logic circuits and enable students to apply it to design a digital system.

COURSE OUTCOMES:

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<td>1</td>
<td>Explain the elements of digital system abstractions such as digital representations of information, digital logic and Boolean algebra</td>
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<td>2</td>
<td>Create an implementation of a combinational logic function described by a truth table using and/or/inv gates/ MUXes</td>
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<td>3</td>
<td>Compare different types of logic families with respect to performance and efficiency</td>
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<tr>
<td>4</td>
<td>Design a sequential logic circuit using the basic building blocks like flip-flops</td>
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<tr>
<td>5</td>
<td>Design and analyze combinational and sequential logic circuits through gate level Verilog models.</td>
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CO-PO-PSO MAPPING:

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<td>Students will learn digital logic and Boolean algebra.</td>
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<td>Students will analyze digital representation of information to decide on the type of circuits.</td>
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<td>Students will learn the working and design of combinatorial circuits.</td>
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<td>ECT203.3-PO2</td>
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<td>Students will analyze the behavior of various logic families to make decisions, on the type of logic families to be chosen, for various applications.</td>
</tr>
<tr>
<td>ECT203.4-PO1</td>
<td>3</td>
<td>Students will learn the working and design of sequential circuits.</td>
</tr>
<tr>
<td>ECT203.4-PO2</td>
<td>3</td>
<td>Students will analyze the requirements and make conclusions on the type of circuits to be designed.</td>
</tr>
<tr>
<td>ECT203.4-PO3</td>
<td>3</td>
<td>Students will perform design of sequential circuits.</td>
</tr>
<tr>
<td>ECT203.5-PO1</td>
<td>3</td>
<td>Students will learn and apply the Knowledge of Logic circuit design to describe its behavior in Verilog HDL.</td>
</tr>
<tr>
<td>ECT203.5-PO2</td>
<td>3</td>
<td>Students will analyze the simulation output to verify the correctness of the HDL model.</td>
</tr>
<tr>
<td>ECT203.5-PO3</td>
<td>3</td>
<td>Students will be able the design HDL models of digital circuits (problems) using Verilog.</td>
</tr>
<tr>
<td>ECT203.5-PO5</td>
<td>3</td>
<td>Students will study and perform programming of Verilog HDL.</td>
</tr>
</tbody>
</table>

### JUSTIFICATION FOR CO-PSO MAPPING

<table>
<thead>
<tr>
<th>MAPPING</th>
<th>LEVEL</th>
<th>JUSTIFICATION</th>
</tr>
</thead>
<tbody>
<tr>
<td>ECT203.1-PSO1</td>
<td>3</td>
<td>Students will learn digital logic and Boolean algebra.</td>
</tr>
<tr>
<td>ECT203.2-PS01</td>
<td>3</td>
<td>Students will learn the working and design of combinatorial circuits.</td>
</tr>
</tbody>
</table>
ECT203.3-PSO1  3  Students will learn the working of different types of logic families.
ECT203.4-PSO1  3  Students will learn the working and design of sequential circuits.
ECT203.5-PSO2  3  Students will learn and apply modern tools such as Verilog HDL to design and analyze logic circuits.

### GAPS IN THE SYLLABUS - TO MEET INDUSTRY/PROFESSION REQUIREMENTS:

<table>
<thead>
<tr>
<th>Sl No</th>
<th>DESCRIPTION</th>
<th>PROPOSED ACTIONS</th>
<th>PO MAPPING</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Familiarization of HDL tools through hands on session</td>
<td>Conduct hands on session for Verilog HDL</td>
<td>PO-5</td>
</tr>
</tbody>
</table>

PROPOSED ACTIONS: TOPICS BEYOND SYLLABUS/ASSIGNMENT/INDUSTRY VISIT/GUEST LECTURER/NPTEL ETC

### TOPICS BEYOND SYLLABUS/ADVANCED TOPICS/DESIGN:

<table>
<thead>
<tr>
<th>Sl No</th>
<th>DESCRIPTION</th>
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<tbody>
<tr>
<td>2</td>
<td>Implementation of digital circuit designs on FPGA</td>
<td>PO-3, PO-5</td>
</tr>
</tbody>
</table>

### WEB SOURCE REFERENCES:

2. [http://www.electronics-tutorials.ws/logic/logic_1.html](http://www.electronics-tutorials.ws/logic/logic_1.html)

### DELIVERY/INSTRUCTIONAL METHODOLOGIES:

- ✓ CHALK & TALK
- ✓ STUD. ASSIGNMENT
- ✓ WEB RESOURCES
- □ LCD/SMART BOARDS
- ✓ STUD. SEMINARS
- □ ADD-ON COURSES

### ASSESSMENT METHODOLOGIES-DIRECT

- □ ASSIGNMENTS  □ STUD. SEMINARS  ✓ TESTS/MODEL EXAMS  ✓ UNIV. EXAMINATION
- ✓ STUD. LAB PRACTICES  ✓ STUD. VIVA  ✓ MINI/MAJOR PROJECTS  □ CERTIFICATIONS
- □ ADD-ON COURSES  □ OTHERS
ASSESSMENT METHODOLOGIES-INDIRECT

☐ ASSESSMENT OF COURSE OUTCOMES (BY FEEDBACK, ONCE)
☐ STUDENT FEEDBACK ON FACULTY
☐ ASSESSMENT OF MINI/MAJOR PROJECTS BY EXT. EXPERTS
☐ OTHERS

Prepared by
Aparna George

Approved by
(HOD)

COURSE PLAN

Subject Code: ECT203: LOGIC CIRCUIT DESIGN

<table>
<thead>
<tr>
<th>Sl.No</th>
<th>Module</th>
<th>Planned Date</th>
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<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>17-Aug-20</td>
<td>Introduction to Logic Circuit Design</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>18-Aug-20</td>
<td>Binary, Decimal Number System</td>
</tr>
<tr>
<td>3</td>
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<td>20-Aug-20</td>
<td>Octal and hexadecimal number systems</td>
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<tr>
<td>4</td>
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<td>21-Aug-20</td>
<td>Methods of base conversions</td>
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<tr>
<td>5</td>
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<td>24-Aug-20</td>
<td>Binary, octal and hexadecimal arithmetic</td>
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<tr>
<td>6</td>
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<td>25-Aug-20</td>
<td>Representation of signed numbers</td>
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<td>7</td>
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<td>27-Aug-20</td>
<td>Representation of signed numbers</td>
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<tr>
<td>8</td>
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<td>07-Sep-20</td>
<td>Binary coded decimal codes; Gray codes; Excess 3 code</td>
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<td>9</td>
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<td>08-Sep-20</td>
<td>Error detection and correction codes - parity check codes</td>
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<td>10</td>
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<td>10-Sep-20</td>
<td>Error detection and correction codes - Hamming code</td>
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<td>11</td>
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<td>Error detection and correction codes - Hamming code</td>
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<tr>
<td>12</td>
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<td>14-Sep-20</td>
<td>Alphanumeric codes: ASCII</td>
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<tr>
<td>13</td>
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<td>15-Sep-20</td>
<td>Verilog basic language elements: identifiers, data objects</td>
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<td>14</td>
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<td>17-Sep-20</td>
<td>Scalar data types, operators</td>
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<tr>
<td>15</td>
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<td>18-Sep-20</td>
<td>Boolean postulates and laws, De-Morgan's Theorems, Principle of Duality</td>
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<td>16</td>
<td>2</td>
<td>22-Sep-20</td>
<td>Logic Functions and Gates</td>
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<td>17</td>
<td>2</td>
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<td>Minimization of Boolean expressions</td>
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<td>18</td>
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<td>25-Sep-20</td>
<td>Sum of Products (SOP), Product of Sums (POS)</td>
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<td>19</td>
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<td>28-Sep-20</td>
<td>Canonical forms, Karnaugh map Minimization</td>
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<td>20</td>
<td>2</td>
<td>29-Sep-20</td>
<td>Gate level modelling in Verilog: Basic gates</td>
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<td>21</td>
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<td>01-Oct-20</td>
<td>Gate level modelling in Verilog: XOR using NAND and NOR</td>
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<td>22</td>
<td>3</td>
<td>12-Oct-20</td>
<td>Combinatorial Logic Systems - Comparators</td>
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<tr>
<td>23</td>
<td>3</td>
<td>13-Oct-20</td>
<td>Multiplexers, Demultiplexers</td>
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<tr>
<td>24</td>
<td>3</td>
<td>15-Oct-20</td>
<td>Encoder, Decoder</td>
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<tr>
<td>25</td>
<td>3</td>
<td>16-Oct-20</td>
<td>Half and Full Adders, Subtractors</td>
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<td>Topic</td>
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<td>Serial and Parallel Adders, BCD Adder</td>
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<tr>
<td>27</td>
<td>Verilog: half adder, full adder</td>
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<tr>
<td>28</td>
<td>Verilog: mux, demux</td>
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<tr>
<td>29</td>
<td>Verilog: decoder, encoder</td>
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<td>30</td>
<td>Sequential Logic Circuits: S-R, JK, T and D FF</td>
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<td>31</td>
<td>Master-Slave JK FF, Edge triggered FF</td>
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<td>32</td>
<td>Conversion of Flipflops, Excitation table and characteristic equation</td>
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<td>33</td>
<td>Ripple and Synchronous counters</td>
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<td>Shift registers-SIPO,SISO,PISO,PIPO</td>
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<td>35</td>
<td>Ring counter and Johnsons counter</td>
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<td>36</td>
<td>Asynchronous counter design</td>
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<td>37</td>
<td>Synchronous counter design</td>
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<td>38</td>
<td>Synchronous counter design</td>
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<td>39</td>
<td>Mod N counter, Random Sequence generator</td>
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<td>40</td>
<td>Modelling sequential logic circuits in Verilog: flipflops</td>
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<td>41</td>
<td>Modelling sequential logic circuits in Verilog: counters</td>
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<tr>
<td>42</td>
<td>Logic families: TTL, ECL, CMOS</td>
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<tr>
<td>43</td>
<td>Electrical characteristics of logic gates – logic levels and noise margins, fan-out</td>
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<tr>
<td>44</td>
<td>propagation delay, transition time, power consumption and power-delay product</td>
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<tr>
<td>45</td>
<td>TTL inverter - circuit description and operation</td>
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<tr>
<td>46</td>
<td>CMOS inverter - circuit description and operation</td>
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<tr>
<td>47</td>
<td>Structure and operations of TTL and CMOS gates</td>
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<tr>
<td>48</td>
<td>NAND in TTL</td>
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<tr>
<td>49</td>
<td>NAND and NOR in CMOS</td>
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</tbody>
</table>

**COURSE PROJECT (ASSIGNMENT QUESTIONS)**

**Perform modelling of following digital circuits using verilog and verify its working**

1. 4:1 MUX, 8:1 MUX
2. 1:4 DEMUX, 1:8 DEMUX
3. Half Adder, Full Adder
4. Half Subtractor, Full Subtractor
5. 2 to 4 decoder, 3 to 8 decoder
6. 4 to 2 encoder, 8 to 3 encoder
7. \( f(A,B,C,D) = \sum(4,5,7,8,9,11,12,13,15) \)
8. \( f(A,B,C,D) = \sum(2,3,5,7,9,11,12,13,14,15) \)
9. 1 bit comparator, 2 bit comparator
10. 3 bit comparator
11. \( f(A,B,C,D) = \pi(0,2,5,7,8,10,13,15) \)
12. \( f(A,B,C,D) = \pi(1,3,4,6,7,9,11,12,14,15) \)
13. Implement Truth Table given below, take the column Y1 (only)
14. Implement Truth Table given below, take the column Y2 (only)
15. Implement Truth Table given below, take the column Y3 (only)
16. Implement Truth Table given below, take the column Y4 (only)
17 Implement Truth Table given below, take the column Y5 (only)
18 Implement Truth Table given below, take the column Y6 (only)
19 SR and D Flipflop
20 JK and T Flipflop
21 4 bit asynchronous/Ripple counter
22 4 bit synchronous counter
23 Ring counter
24 Johnson counter
25 Universal Shift register

<table>
<thead>
<tr>
<th>Input</th>
<th>Output</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Y1</td>
</tr>
<tr>
<td>A</td>
<td>B</td>
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<td>0</td>
</tr>
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</table>

**TUTORIAL QUESTIONS**

1. A function is defined as \( f(a,b,c,d) = a' + a'c + c' + a'd + a'b'c + a'bc' \)
   1) Express the function in standard SOP form
   2) Simplify the function using K-map and implement using NAND gates only
2. Express \( f = AB + AC + CD + ABC + ABC \) in standard SOP form
3. Using K map simplify the SOP function and realize it using logic gates
   \( f(a,b,c,d) = \Sigma m(0,1,2,4,5,6), d = \Sigma (3,7,14,15) \)
4. Obtain the minimal SOP expression for \( \Sigma m(2,3,5,7,9,11,12,13,14,15) \) and implement it using NAND gate.
5. Minimize the following logic function using k-map and realize using NAND gates alone.
   \( F(A,B,C,D) = \Sigma m(0,3,5,8,9,11,15) + d(2,3) \)
6. A four variable Boolean function is given as \( F = ABC + AC'D + BCD \) where \( ABC'D' + AB'CD + A'B'CD \) are don't cares. Use karnaugh map to find the minimal SOP expression for \( F \).
7. Draw the gate level circuit diagram for 1 to 4 demultiplexer. For the Boolean function 
   \( F = (A+B)(A+C)(B+C) \). Show how it can be implemented using a 1:8 demultiplexer and one 
or more gates.

8. A function is defined as
   \[ F(a,b,c,d) = a'b + a'c + c' + a'd + a'b'c + a'bc'. \]
   Implement using a single 8:1 MUX.

9. Explain the operation of an 8x1 multiplexer and implement the following using an 8X1 
multiplexer:
   \[ F(A,B,C,D) = \sum m(0,1,3,5,6,7,8,9,11,13,14) \]

10. Design & realize a combinational circuit to compare two 3 bit numbers A (A2A1A0) and 
   B (B2B1B0) as inputs and "AGT"(A>B), "AEQ"(A=B) and "ALT"(A<B) are the outputs.

11. Design a logic circuit that produces a HIGH output whenever a 3 bit binary number 
    A2A1A0 greater than 001 and less than 110 is applied as input (A2 is MSB).

12. A logic circuit has four inputs A, B, C and D. A four bit input is fed with A as MSB and D 
as LSB. Design and implement a circuit such that the output is one when the input is 
more than or equal to decimal 6.

13. Realize the following: i) T flip flop using SR flip flop ii) JK flip flop using D flip flop.

14. Realize a T flip flop using NAND gates and explain the operation with truth table, 
    excitation table and characteristic equation.

15. Convert SR flip flop into JK flip flop.
ECT 205 NETWORK THEORY
Course Information Sheet

Programme: Electronics & Communication Engineering
Degree: B. Tech

Course: Network Theory
Semester: 3
Credits: 2
Course Code: ECT205
Regulation: 2019
Course Type: Core
Course Area/Domain: Electronic Circuits
Contact Hours: 4 hrs.

Corresponding Lab Course Code (If Any): Nil
Lab Course Name: Nil

Syllabus:

Module 1: Mesh and Node Analysis
Mesh and node analysis of network containing independent and dependent sources. Supermesh and Supernode analysis. Steady-state AC analysis using Mesh and Node analysis.

Module 2: Network Theorems
Thevenin's theorem, Norton's theorem, Superposition theorem, Reciprocity theorem, Maximum power transfer theorem. (applied to both dc and ac circuits having dependent source).

Module 3: Application of Laplace Transforms

Module 4: Network functions

Module 5: Two port network Parameters

Text/Reference Books:

<table>
<thead>
<tr>
<th>T/R</th>
<th>Book Title/Authors/Publication</th>
</tr>
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COURSE PRE-REQUISITES:

<table>
<thead>
<tr>
<th>C.CODE</th>
<th>COURSE NAME</th>
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<tbody>
<tr>
<td>EST130</td>
<td>Basics of Electrical and Electronics Engineering</td>
</tr>
<tr>
<td>MAT102</td>
<td>Vector Calculus, Differential Equations and Transforms (Laplace Transform)</td>
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</table>

COURSE OBJECTIVES:

1. To familiarize students with analysis of linear time invariant electronic circuits.

COURSE OUTCOMES:

<table>
<thead>
<tr>
<th>Sl. No</th>
<th>DESCRIPTION</th>
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<tbody>
<tr>
<td>1</td>
<td>Apply Mesh / Node analysis or Network Theorems to obtain steady state response of the linear time invariant networks.</td>
</tr>
<tr>
<td>2</td>
<td>Apply Laplace Transforms to determine the transient behaviour of RLC networks.</td>
</tr>
<tr>
<td>3</td>
<td>Apply Network functions and Network Parameters to analyse the single port and two port networks.</td>
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CO-PO-PSO MAPPING

<table>
<thead>
<tr>
<th>CO No.</th>
<th>Programme Outcomes (POs)</th>
<th>Programme-specific Outcomes (PSOs)</th>
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JUSTIFICATION FOR THE CORRELATION LEVEL ASSIGNED IN EACH CELL OF THE TABLE

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<th>PO2</th>
<th>PO12</th>
<th>PSO1</th>
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<tbody>
<tr>
<td><strong>CO1</strong></td>
<td>Basics for real-world electrical network &amp; electronic circuit analysis</td>
<td>Techniques for circuit analysis under different conditions like dependent sources, dc circuits, ac circuits etc...</td>
<td>Lays foundation for more advanced topics in network theory and analysis</td>
<td>Principles &amp; techniques learnt can be extended to many future courses like Electronic circuits, Control theory etc...</td>
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<tr>
<td><strong>CO2</strong></td>
<td>Applying Laplace transform covered in mathematics to a specific engineering problem</td>
<td>Using Laplace transforms to perform frequency domain analysis of circuits and extending analysis to excitations beyond sinusoids</td>
<td>Provides scope for more advanced analysis of circuits in the frequency domain</td>
<td>Principles &amp; techniques learnt can be extended to many future courses like Electronic circuits, Control theory etc...</td>
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<tr>
<td><strong>CO3</strong></td>
<td>Network modelling and generalization.</td>
<td>Expressing networks using different parameter sets and simplifying analysis based on problem at hand</td>
<td>Leads to more complex topics like circuit stability analysis, transmission line concepts etc...</td>
<td>Principles &amp; techniques learnt can be extended to many future courses like Electronic circuits, Control theory etc...</td>
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</table>

GAPS IN THE SYLLABUS - TO MEET INDUSTRY/PROFESSION REQUIREMENTS:

<table>
<thead>
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<th>SI No</th>
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<th>PROPOSED ACTIONS</th>
<th>PO MAPPING</th>
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<tbody>
<tr>
<td>1</td>
<td>System modeling and analysis-checking stability and energy conservation.</td>
<td>Assignments on Laplace Transform, Z transform etc</td>
<td>P01,P02,P03,P04,P05, P012</td>
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<tr>
<td>2</td>
<td>Solving first order linear homogeneous and non-homogeneous equations</td>
<td>Assignment (Mathematics)</td>
<td>P01,P02,P03,P04,P05, P012</td>
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PROPOSED ACTIONS: TOPICS BEYOND SYLLABUS/ASSIGNMENT/INDUSTRY VISIT/GUEST LECTURER/NPTEL ETC

TOPICS BEYOND SYLLABUS/ADVANCED TOPICS/DESIGN:

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</table>
### Introduction to PSpice

1. Introduction to PSpice
2. MATLAB examples

### WEB SOURCE REFERENCES:
1. https://nptel.ac.in/courses/108/106/108106075/
4. https://www.youtube.com/watch?v=luJMI37-nso

### DELIVERY/INSTRUCTIONAL METHODOLOGIES:

<table>
<thead>
<tr>
<th>Methodology</th>
<th>Chalk &amp; Talk</th>
<th>Stud. Assignment</th>
<th>Web Resources</th>
<th>LCD/Smart Boards</th>
<th>Stud. Seminars</th>
<th>Add-On Courses</th>
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<tbody>
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### ASSESSMENT METHODOLOGIES-DIRECT

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### ASSESSMENT METHODOLOGIES-INDIRECT

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Prepared by

Ms. Liza Annie Joseph

Approved by

Dr. Hari C.V.
(HoD)
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<td>applied to both dc and ac circuits having dependent source with examples</td>
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<td>Transient analysis of RL, RC, and RLC networks with exponential and sinusoidal inputs</td>
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<td>33</td>
<td>4</td>
<td>Significance of Poles and Zeros of network functions, Time domain response from pole zero plot</td>
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</table>
### ASSIGNMENT 1

1. Derive the expression for the equivalent capacitance when three capacitors are connected
   a. in series and
   b. in parallel

2. Derive the expression for the equivalent inductance when three inductors are connected
   a. in series and
   b. in parallel
1. Find the currents in the three meshes of the network shown.

2. For the network shown, find the power supplied by the dependent voltage source using mesh analysis.

3. For the network shown, find the current through 8Ω resistor.
4. Find the current through 8Ω resistor in the following circuit using mesh analysis.

5. For the network shown, find the voltage $V_x$.

6. For the n/w shown, find voltages $V_1$ and $V_2$. 
7. Find the voltage across the 4Ω resistor shown in the circuit using source transformations.

8. Find the power delivered by the 5Ω current source using nodal analysis method.
1. Find the value of $V_0$ such that no current flows through $R_0$. 

2. Find the voltage across $i_{inductor}$. Also find the power dissipated across $i_{inductor}$. 

3. Find the voltage across $R_{02}$ resistor using mesh analysis.
4. Find current $I$ using node analysis.

\[ I = -3.43 \, \text{A} \]

5. Determine the value of $V_2$ such that the current through the impedance $(3+j4) \, \Omega$ is zero.

\[ V_2 = 16.97 \, \text{V} \]

6. Using Thévenin's theorem, find the power dissipated across 24 \, \Omega \, \text{resistor}.

\[ \text{P} = 96 \, \text{W} \]
1. Using superposition theorem, find the voltage across \((2 + j5)\)Ω impedance for the m/o shown.

2. Find the voltage drop across the capacitor for the m/o shown.
3. Determine the voltage across 10 Ohm, connected between the terminals A and B using superposition theorem.

4. Find the current in each resistor using the superposition theorem.

5. Using superposition theorem, find the value of current through the capacitor.
6. Find the current in the 6Ω resistor, using Superposition.

7. Find the Thévenin's equivalent of the circuit shown.

8. Find the value of load and maximum power delivered to load.
ECT205

Tutorial 2 / Assignment 5 6.11.20

1) For the n/o shown, find the current \( i(t) \) when switch is changed from position 1 to 2 at \( t=0 \).

\[
\begin{align*}
\text{500V} & \quad 40 \Omega & \quad 60 \Omega \\
& \quad 0.4 \text{H} & \\
& \quad 10 \Omega & \\
\end{align*}
\]

2) The switch is opened at \( t=0 \) after steady state is achieved. Find the expression for the transient current \( i(t) \).

\[
\begin{align*}
\text{50V} & \quad 50 \Omega & \quad 20 \Omega \\
& \quad 3.0 \text{H} & \quad 3.0 \text{H} \\
\end{align*}
\]
MCN 201 SUSTAINABLE ENGINEERING
COURSE INFORMATION SHEET

<table>
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<td>CORRESPONDING LAB COURSE CODE (IF ANY): NIL</td>
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COURSE OBJECTIVES

Objective of this course is to inculcate in students an awareness of environmental issues and the global initiatives towards attaining sustainability. The student should realize the potential of technology in bringing in sustainable practices.

COURSE PRE-REQUISITES:

NIL

SYLLABUS:


Module 1
Sustainability: Introduction, concept, evolution of the concept; Social, environmental and economic sustainability concepts; Sustainable development, Nexus between Technology and Sustainable development; Millennium Development Goals (MDGs) and Sustainable Development Goals (SDGs), Clean Development Mechanism (CDM).

Module 2
Environmental Pollution: Air Pollution and its effects, Water pollution and its sources, Zero waste concept and 3 R concepts in solid waste management; Greenhouse effect, Global warming, Climate change, Ozone layer depletion, Carbon credits, carbon trading and carbon foot print, legal provisions for environmental protection.

Module 3

Module 4
Resources and its utilisation: Basic concepts of Conventional and non-conventional energy, General idea about solar energy, Fuel cells, Wind energy, Small hydro plants, bio-fuels, Energy derived from oceans and Geothermal energy.

Module 5
Sustainability practices: Basic concept of sustainable habitat, Methods for increasing energy efficiency in buildings, Green Engineering, Sustainable Urbanisation, Sustainable cities, Sustainable transport.
**TEXT/REFERENCE BOOKS:**

<table>
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<th>T/R</th>
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<tbody>
<tr>
<td>T1</td>
<td>Sustainability Engineering: Concepts, Design and Case Studies</td>
<td>Allen, D. T. and Shonnard, D. R.</td>
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<td></td>
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<td>Prentice Hall.</td>
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<td>T2</td>
<td>Engineering applications in sustainable design and development</td>
<td>Bradley, A.S.; Adebayo, A.O.; Maria, P.</td>
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<td>Environment Impact Assessment Guidelines, Notification of Government</td>
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<td>of India, 2006</td>
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<td>Mackenthun, K.M.</td>
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<td>T6</td>
<td>Systems Analysis for Sustainable Engineering: Theory and Applications</td>
<td>Ni bin Chang</td>
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<td>T7</td>
<td>Renewable Energy Resources</td>
<td>Twidell, J.W. and Weir, A. D.</td>
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<td>T8</td>
<td>Green Technology - An approach for sustainable environment</td>
<td>Purohit, S. S.</td>
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**WEB SOURCE REFERENCES**

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<td>CO1</td>
<td>Understand the relevance and the concept of sustainability and the global initiatives in this direction</td>
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<tr>
<td>CO2</td>
<td>Explain the different types of environmental pollution problems and their sustainable solutions</td>
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<tr>
<td>CO3</td>
<td>Discuss the environmental regulations and standards</td>
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<td>CO4</td>
<td>Outline the concepts related to conventional and non-conventional energy</td>
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<tr>
<td>CO5</td>
<td>Demonstrate the broad perspective of sustainable practices by utilizing engineering knowledge and principles</td>
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**COURSE OUTCOMES**

- [http://www.investopedia.com/terms/c/carbon_credit.asp#ixzz3fTars0xW](http://www.investopedia.com/terms/c/carbon_credit.asp#ixzz3fTars0xW)
- [http://www.unwater.org/fileadmin/user_upload/unwater_new/docs/UN-Water_Analytical_Brief_Wastewater_Management.pdf](http://www.unwater.org/fileadmin/user_upload/unwater_new/docs/UN-Water_Analytical_Brief_Wastewater_Management.pdf)
- [http://ems.iema.net/faq](http://ems.iema.net/faq)
- [http://ems.iema.net/faq](http://ems.iema.net/faq)
## CO-PO AND CO-PSO MAPPING

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## JUSTIFICATIONS FOR CO – PO MAPPING

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<td>M</td>
<td>Sustainable concept to be clearly understood in order to take responsibilities related to the society.</td>
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<td><strong>CO.1 – PO.7</strong></td>
<td>H</td>
<td>Sustainable concept requires for environmentally protective, socially equitable and economically viable engineering solutions.</td>
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<td><strong>CO.1 – PO.12</strong></td>
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<td>Taking initiatives of the global movement of sustainable growth continuing</td>
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<td>Sustainability aspects in the technical training helps the students to make global initiatives.</td>
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<td><strong>CO.2 – PO.6</strong></td>
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<td>Need of concertizing the society about adopting the measures to tackle environmental problems</td>
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<td><strong>CO.2 – PO.7</strong></td>
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<td>Understanding complex engineering problem in environmental pollution</td>
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<td><strong>CO.2 – PO.12</strong></td>
<td>M</td>
<td>Contributing to the global effort of meeting environmental issues</td>
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<td><strong>CO.3 – PO.6</strong></td>
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<td>Societal responsibility of maintaining the environmental regulations and standard properly</td>
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<tr>
<td><strong>CO.3 – PO.7</strong></td>
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<td>Compliance with environmental standards in system designing</td>
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<tr>
<td><strong>CO.3 – PO.12</strong></td>
<td>M</td>
<td>Relating the current urbanisation issues to the science of ecology.</td>
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<td><strong>CO.4 – PO.6</strong></td>
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<td>Adopting ecofriendly energy sources and developing techniques and making them affordable to the society</td>
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<td><strong>CO.4 – PO.7</strong></td>
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<td>Class on science of eco systems and the role of eco systems in sustainable engineering.</td>
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<tr>
<td><strong>CO.4 – PO.12</strong></td>
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<td>High priority for sustainability in energy utilization techniques- Global vision and updation.</td>
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<td><strong>CO.5 – PO6</strong></td>
<td>M</td>
<td>Leadership in using new technologies for the benefits of the Society considering sustainable principles.</td>
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## CO.5-PO7
- **H**
  - Broader vision of environment protection in developing all sorts of engineering systems

## CO.5-PO12
- **M**
  - Global aspects of efficient sustainable systems adaptable to the current particular situations needed

## CO-5-PSO3
- **H**
  - Sustainable concepts and practices make the student for the future leaderships industrial developments and involvements

### Gaps in the Syllabus, to meet Industry / Profession Requirements

<table>
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<th>Proposed Actions</th>
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<td>Role of engineers in sustainable development</td>
<td>Discuss in class</td>
<td>7, 11</td>
</tr>
<tr>
<td>2</td>
<td>A movement to environmental spirituality</td>
<td>Class discussion</td>
<td>6, 7, 8, 12</td>
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<tr>
<td>3</td>
<td>Scope of case studies of actual issues of renewable energy implementations</td>
<td>Assignment on particular case studies</td>
<td>7, 11</td>
</tr>
</tbody>
</table>

### TOPICS BEYOND SYLLABUS / ADVANCED TOPICS / DESIGN

<table>
<thead>
<tr>
<th>Sl. No.</th>
<th>Description</th>
<th>Proposed actions</th>
<th>Relevance with POs</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>General Preparation for sensing our own environment in current time</td>
<td>Video making of my understanding about sustainability sensing my own environment</td>
<td>6, 7, 8</td>
</tr>
<tr>
<td>2</td>
<td>Studies of Sustainable development vs Pandemic Outbreak</td>
<td>Assignment for self-study</td>
<td>5, 6, 7, 11</td>
</tr>
</tbody>
</table>

### DELIVERY/INSTRUCTIONAL METHODOLOGIES:

<table>
<thead>
<tr>
<th>Methodology</th>
<th>Description</th>
<th>Action/Comment</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>CHALK &amp; TALK</strong> - Some of the classes</td>
<td>STUD. projects- some√</td>
<td>□ WEB RESOURCES</td>
</tr>
<tr>
<td><strong>IT Enabled / PPT – Many of the classes√</strong></td>
<td>STUD. SEMINARS / Presentations – for a few students</td>
<td>DISCUSSIONS/presentationsrandomly√</td>
</tr>
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</table>
ASSESSMENT METHODOLOGIES-DIRECT:

☐ Assignments ✓ ☐ STUD. SEMINARS ☐ TESTS/MODEL EXAMS ✓ ☐ UNIV. EXAMINATION ✓

☐ STUD. LAB PRACTICES ☐ STUD. VIVA ☐ MINI/MAJOR PROJECTS ☐ CERTIFICATIONS

☐ ADD-ON COURSES ☐ OTHERS-

ASSESSMENT METHODOLOGIES-INDIRECT:

☐ ASSESSMENT OF COURSE OUTCOMES (BY FEEDBACK) ✓ ☐ Through interactions with students ✓

Prepared by
Fr. Thomas PJ

Approved by
Dr. Hari C V (HOD, AEI)

COURSE PLAN

<table>
<thead>
<tr>
<th>Sl.No</th>
<th>Module</th>
<th>Planned</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>Introduction, concept, evolution of the concept</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>Social, environmental and economic sustainability concepts</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>Sustainable development, Nexus between Technology and Sustainable development</td>
</tr>
<tr>
<td>4</td>
<td>1</td>
<td>Millennium Development Goals (MDGs) and Sustainable Development Goals (SDGs)</td>
</tr>
<tr>
<td>5</td>
<td>1</td>
<td>Clean Development Mechanism (CDM)</td>
</tr>
<tr>
<td>6</td>
<td>2</td>
<td>Air Pollution and its effects</td>
</tr>
<tr>
<td>7</td>
<td>2</td>
<td>Water pollution and its sources</td>
</tr>
<tr>
<td>8</td>
<td>2</td>
<td>Zero waste concept and 3 R concepts in solid waste management</td>
</tr>
<tr>
<td>9</td>
<td>2</td>
<td>Greenhouse effect, Global warming, Climate change, Ozone layer depletion</td>
</tr>
<tr>
<td>10</td>
<td>2</td>
<td>Carbon credits, carbon trading and carbon foot print.</td>
</tr>
<tr>
<td>11</td>
<td>2</td>
<td>Legal provisions for environmental protection</td>
</tr>
<tr>
<td>12</td>
<td>3</td>
<td>Environmental management standards</td>
</tr>
<tr>
<td>13</td>
<td>3</td>
<td>ISO 14001:2015 frame work and benefits</td>
</tr>
<tr>
<td>14</td>
<td>3</td>
<td>Scope and Goal of Life Cycle Analysis (LCA)</td>
</tr>
<tr>
<td>15</td>
<td>3</td>
<td>Circular economy, Bio-mimicking</td>
</tr>
<tr>
<td>16</td>
<td>3</td>
<td>Environment Impact Assessment (EIA)</td>
</tr>
</tbody>
</table>
### ASSIGNMENT 1

**Questions**

1. Make a video on “My Sustainability Thoughts by Sensing my own Environment”
2. Write about Sustainable Development VS Pandemic Outbreak

### ASSIGNMENT 2

**Questions**

1. Make a video on Non-Conventional Energy Technologies Implemented in India.
ECL 201 SCIENTIFIC COMPUTING LABORATORY
# COURSE INFORMATION SHEET

<table>
<thead>
<tr>
<th>PROGRAMME: APPLIED ELECTRONICS AND INSTRUMENTATION</th>
<th>DEGREE: BTECH</th>
</tr>
</thead>
<tbody>
<tr>
<td>COURSE: SCIENTIFIC COMPUTING LAB</td>
<td>SEMESTER: 3 CREDITS: 2</td>
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<tr>
<td>COURSE CODE: ECL201</td>
<td>COURSE TYPE: CORE</td>
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<tr>
<td>REGULATION: 2019</td>
<td>CONTACT HOURS: 0+0+3 (LAB) hours/Week.</td>
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<tr>
<td>COURSE AREA/DOMAIN: ELECTRONICS</td>
<td>THEORY COURSE NAME: NIL</td>
</tr>
<tr>
<td>CORRESPONDING THEORY COURSE CODE (IF ANY): NIL</td>
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**SYLLABUS:**

<table>
<thead>
<tr>
<th>SL.NO</th>
<th>DETAILS</th>
<th>HOURS</th>
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<tbody>
<tr>
<td>1</td>
<td>Familiarization of the Computing Tool</td>
<td>3</td>
</tr>
<tr>
<td>2</td>
<td>Familiarization of Scientific Computing</td>
<td>3</td>
</tr>
<tr>
<td>3</td>
<td>Realization of Arrays and Matrices</td>
<td>3</td>
</tr>
<tr>
<td>4</td>
<td>Numerical Differentiation and Integration</td>
<td>3</td>
</tr>
<tr>
<td>5</td>
<td>Solution of Ordinary Differential Equations</td>
<td>3</td>
</tr>
<tr>
<td>6</td>
<td>Simple Data Visualization</td>
<td>3</td>
</tr>
<tr>
<td>7</td>
<td>Simple Data Analysis with Spreadsheets</td>
<td>3</td>
</tr>
<tr>
<td>8</td>
<td>Convergence of Fourier Series</td>
<td>3</td>
</tr>
<tr>
<td>9</td>
<td>Coin Toss and the Level Crossing Problem</td>
<td>3</td>
</tr>
</tbody>
</table>

**TOTAL HOURS** 27

**TEXT/REFERENCE BOOKS:**

1. DIGITAL SIGNAL PROCESSING USING MATLAB by Vinay K. Lingle, John G. Proakis.

**COURSE PRE-REQUISITES:**

<table>
<thead>
<tr>
<th>C.CODE</th>
<th>COURSE NAME</th>
<th>DESCRIPTION</th>
<th>SEM</th>
</tr>
</thead>
<tbody>
<tr>
<td>MAT 101</td>
<td>Linear Algebra and Calculus</td>
<td>Theory</td>
<td>1</td>
</tr>
<tr>
<td>MAT 102</td>
<td>Vector Calculus, Differential Equations and Transforms</td>
<td>Theory</td>
<td>2</td>
</tr>
</tbody>
</table>

**COURSE OBJECTIVES:**

1. To translate the mathematical concepts into system design.
2. To familiarize with computing tools such as Matlab and Python
3. The experiments will lay the foundation for future labs such as DSPlab.

**COURSE OUTCOMES:**
<table>
<thead>
<tr>
<th>Sl. No.</th>
<th>DESCRIPTION</th>
<th>Bloom Taxonomy Level</th>
</tr>
</thead>
<tbody>
<tr>
<td>CO1</td>
<td>Describe the needs and requirements of scientific computing and to familiarize one programming language for scientific computing and data visualization.</td>
<td>Knowledge (Level 1)</td>
</tr>
<tr>
<td>CO2</td>
<td>Approximate an array/matrix with matrix decomposition.</td>
<td>Understand and Apply (Level 2 &amp; 3)</td>
</tr>
<tr>
<td>CO3</td>
<td>Implement numerical integration and differentiation.</td>
<td>Evaluate (Level 5)</td>
</tr>
<tr>
<td>CO4</td>
<td>Solve ordinary differential equations for engineering applications</td>
<td>Create (Level 6)</td>
</tr>
<tr>
<td>CO5</td>
<td>Compute with exported data from instruments</td>
<td>Apply &amp; Analyze (Level 3 &amp; 4)</td>
</tr>
<tr>
<td>CO6</td>
<td>Realize how periodic functions are constituted by sinusoids</td>
<td>Analyze (Level 4)</td>
</tr>
<tr>
<td>CO7</td>
<td>Simulate random processes and understand their statistics.</td>
<td>Apply &amp; Analyze (Level 3 &amp; 4)</td>
</tr>
</tbody>
</table>

CO – PO MAPPING

<table>
<thead>
<tr>
<th>PO1</th>
<th>PO2</th>
<th>PO3</th>
<th>PO4</th>
<th>PO5</th>
<th>PO6</th>
<th>PO7</th>
<th>PO8</th>
<th>PO9</th>
<th>PO10</th>
<th>PO11</th>
<th>PO12</th>
<th>PSO1</th>
<th>PSO2</th>
<th>PSO3</th>
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</thead>
<tbody>
<tr>
<td>CO1</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>2</td>
<td>3</td>
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<tr>
<td>CO2</td>
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<td>0</td>
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</tr>
<tr>
<td>CO3</td>
<td>3</td>
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<td>1</td>
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<tr>
<td>CO4</td>
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<td>3</td>
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<tr>
<td>CO5</td>
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<tr>
<td>CO6</td>
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<td>3</td>
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<td>2</td>
<td>3</td>
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<td>1</td>
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<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>CO7</td>
<td>3</td>
<td>3</td>
<td>2</td>
<td>2</td>
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<td>1</td>
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</table>

GAPS IN THE SYLLABUS - TO MEET INDUSTRY/PROFESSION REQUIREMENTS:

<table>
<thead>
<tr>
<th>SNO</th>
<th>DESCRIPTION</th>
<th>PROPOSED ACTIONS</th>
<th>RELEVANCE WITH POs</th>
<th>RELEVANCE WITH PSOs</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Linear Convolution, circular convolution, modulation techniques</td>
<td>Giving extra lab classes</td>
<td>PO1, PO2</td>
<td>PSO3</td>
</tr>
</tbody>
</table>

PROPOSED ACTIONS: TOPICS BEYOND SYLLABUS/ASSIGNMENT/INDUSTRY VISIT/GUEST LECTURER/NPTEL ETC
TOPICS BEYOND SYLLABUS/ADVANCED TOPICS/DESIGN:
<table>
<thead>
<tr>
<th>SNO</th>
<th>DESCRIPTION</th>
<th>PROPOSED ACTIONS</th>
<th>RELEVANCE WITH POs</th>
<th>RELEVANCE WITH PSOs</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Study of OMAP L138 Development kit</td>
<td>Extra Lab</td>
<td>PO1, PO2</td>
<td>PSO1, PSO3</td>
</tr>
</tbody>
</table>

**WEB SOURCE REFERENCES:**

1. [http://media.sakshat.ac.in/NPTEL-IIT-Videos/default.aspx](http://media.sakshat.ac.in/NPTEL-IIT-Videos/default.aspx)

**DELIVERY/INSTRUCTIONAL METHODOLOGIES:**

- ☐ **CHALK & TALK**
- ☐ **STUD. ASSIGNMENT**
- ☐ **WEB RESOURCES**
- ☐ **LCD/SMART BOARDS**
- ☐ **STUD. SEMINARS**
- ☐ **ADD-ON COURSES**

**ASSESSMENT METHODOLOGIES-DIRECT**

- ☐ **ASSIGNMENTS**
- ☐ **STUD. SEMINARS**
- ☐ **TESTS/MODEL EXAMS**
- ☐ **UNIV. EXAMINATION**
- ☐ **STUD. LAB PRACTICES**
- ☐ **STUD. VIVA**
- ☐ **MINI/MAJOR PROJECTS**
- ☐ **CERTIFICATIONS**
- ☐ **ADD-ON COURSES**
- ☐ **OTHERS**

**ASSESSMENT METHODOLOGIES-INDIRECT**

- ☐ **ASSESSMENT OF COURSE OUTCOMES (BY FEEDBACK, ONCE)**
- ☐ **STUDENT FEEDBACK ON FACULTY (TWICE)**
- ☐ **ASSESSMENT OF MINI/MAJOR PROJECTS BY EXT. EXPERTS**
- ☐ **OTHERS**

Prepared by
Dr. Poornima S
(Faculty)

Approved by
Dr. Hari C.V.
(HOD)
COURSE PLAN

<table>
<thead>
<tr>
<th>SL.NO</th>
<th>MODULE</th>
<th>PLANNED</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>Familiarization of the Computing Tool</td>
</tr>
<tr>
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<td>1</td>
<td>Numerical Differentiation and Integration</td>
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<td>5</td>
<td>1</td>
<td>Solution of Ordinary Differential Equations</td>
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<tr>
<td>6</td>
<td>1</td>
<td>Simple Data Visualization</td>
</tr>
<tr>
<td>7</td>
<td>1</td>
<td>Simple Data Analysis with Spreadsheets</td>
</tr>
<tr>
<td>8</td>
<td>1</td>
<td>Convergence of Fourier Series, Coin Toss and the Level Crossing Problem</td>
</tr>
<tr>
<td>9</td>
<td>1</td>
<td>Test</td>
</tr>
</tbody>
</table>

LAB CYCLE/ LIST OF EXPERIMENTS

1. Familiarization of the Computing Tool
   a. Familiarization of a programming language – MATLAB
   b. Familiarization of data types in the language used.
   c. Familiarization of the syntax of different conditional statements.
   d. Basic syntax and execution of small scripts
      i. Write a program to compute the sum of the digits of an integer.
      ii. Write a program to compute the factorial of an integer.

2. Familiarization of Scientific Computing
   a. Functions with examples
      i. Write a function to print the first N Fibonacci numbers and its sum
      ii. Write a function to print the first N even numbers and its sum.
      iii. Write a function to print the first N odd numbers and its sum.
      iv. Write a function to compute the factorial of an integer.
   b. Basic arithmetic functions such as abs, sine, real, imag, complex, sinc etc. using built in modules.
   c. Vectorized computing without loops for fast scientific applications.

3. Realization of Arrays and Matrices
   a. Write a function to compute the eigen values of a real valued valued matrix (say 5 x 5). Run this code. Plot the eigen values and understand their variation.
   b. Write a function to approximate a 5 x 5 matrix using its first 3 eigen values. Run the code and compute the absolute square error in the approximation.

4. Numerical Differentiation and Integration
a. Realize the functions \( \sin t, \cos t, \sinh t \) and \( \cosh t \) for the vector \( t = [0, 10] \) with increment 0.01
b. Realize the function \( f(t) = 4t^3 + 3 \) and plot it for the vector \( t = [-5; 5] \) with increment 0.01
c. Write and execute a function to return the value of \( \int_{-3}^{3} e^{-|t|} dt \)

5. Solution of Ordinary Differential Equations
   a. Solve the first order differential equation
      \[
      \frac{dx}{dt} + 2x = 0
      \]
      with the initial condition \( x(0) = 1 \)
   b. Write and execute a function to return the numerical solution of
      \[
      \frac{d^2x}{dt^2} + 4 \frac{dx}{dt} + 2x = e^{-t} \cos t
      \]

6. Simple Data Visualization
   a. Draw stem plots, line plots, box plots, bar plots and scatter plots with random data.
   b. Plot the histogram of a random data.
   c. Realize a vector \( t = [-10; 10] \) with increment 0.01 as an array.
   d. Write and execute a program to display random data in two dimensions as continuous and discrete plots.

7. Simple Data Analysis with Spreadsheets
   a. Compute the mean and standard deviation of the signal. Plot its histogram with an appropriate bin size.

8. Convergence of Fourier Series
   a. The experiment aims to understand the lack of convergence of Fourier series.
   b. Write the Fourier series of a triangular signal. Compute this sum for 10 and 50 terms respectively. Plot both signals on the same GUI.

9. Coin Toss and the Level Crossing Problem
   a. Simulate a coin toss that maps a head as 1 and tail as 0.
   b. Write and execute a function to toss three fair coins simultaneously. Compute the probability of getting exactly two heads for 100 and 1000 number of tosses

LAB QUESTIONS

1. Write a MATLAB program to find the factorial of a number.

2. Write a MATLAB program to find the sum of first \( N \) numbers

3. Write a program to compute the sum of the digits of an integer.

4. Write a MATLAB program to print the first \( N \) even numbers and its sum.

5. Write a MATLAB program to print the first \( N \) odd numbers and its sum

6. Realize the functions \( \sin t, \cos t, \sinh t \) and \( \cosh t \) for the vector \( t = [0, 10] \) with increment 0.01
7. Realize the function \( f(t) = 4t^3 + 3 \) and plot it for the vector \( t = [-5; 5] \) with increment 0.01.

8. Write a MATLAB program to compute the value of the function \( f(x) = \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-\frac{x^2}{2}} dx \).

9. Write a MATLAB program to find the integral of the function \( f(t) = 4t^2 + 3t + 2 \), over the limit \( t = -2 \) to \( 2 \).

10. Write a MATLAB program to compute the derivative of the following functions: \( \sin t, \cos t, \sinh t \) and \( \cosh t \) with respect to \( t \), and plot the graphs.

**SCIENTIFIC COMPUTING LAB ASSIGNMENT QUESTIONS – 26/09/2020**

1. Write a function to print the first \( N \) Fibonacci numbers and its sum.
2. Write a function to print the first \( N \) even numbers and its sum.
3. Write a function to print the first \( N \) odd numbers and its sum.
4. Write a function to compute the factorial of an integer.
5. Write a program to compute the sum of the digits of an integer.
ECL 203 LOGIC DESIGN LAB
# COURSE INFORMATION SHEET

<table>
<thead>
<tr>
<th>PROGRAMME: APPLIED ELECTRONICS AND INSTRUMENTATION</th>
<th>DEGREE: B.TECH</th>
</tr>
</thead>
<tbody>
<tr>
<td>COURSE: Logic Design Lab</td>
<td>SEMESTER: 3</td>
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<tr>
<td>COURSE CODE: ECL203</td>
<td>CREDITS: 2</td>
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<td>REGULATION: 2019</td>
<td>COURSE TYPE: CORE</td>
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<td>COURSE AREA/DOMAIN: Digital Electronics</td>
<td>CONTACT HOURS: 2 hrs.</td>
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<tr>
<td>CORRESPONDING LAB COURSE CODE (IF ANY):</td>
<td>LAB COURSE NAME: Nil</td>
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</table>

**SYLLABUS:**

<table>
<thead>
<tr>
<th>Unit</th>
<th>Details</th>
<th>Hrs</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>It is compulsory to conduct a minimum of 5 experiments from Part A and a minimum of 5 experiments from Part B.</td>
<td></td>
</tr>
<tr>
<td></td>
<td><strong>Part A (Any 5)</strong></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>The following experiments can be conducted on breadboard or trainer kits. 1.</td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>Realization of functions using basic and universal gates (SOP and POS forms).</td>
<td>3 hr.</td>
</tr>
<tr>
<td>3</td>
<td>Design and Realization of half/full adder and subtractor using basic gates and universal gates.</td>
<td>3 hr.</td>
</tr>
<tr>
<td>4</td>
<td>4 bit adder/subtractor and BCD adder using 7483.</td>
<td>3 hr.</td>
</tr>
<tr>
<td>5</td>
<td>Study of Flip Flops: S-R, D, T, JK and Master Slave JK FF using NAND gates.</td>
<td>3 hr.</td>
</tr>
<tr>
<td>6</td>
<td>Asynchronous Counter:3 bit up/down counter</td>
<td>3 hr.</td>
</tr>
<tr>
<td>7</td>
<td>Asynchronous Counter: Realization of Mod N counter</td>
<td>3 hr.</td>
</tr>
<tr>
<td>8</td>
<td>Synchronous Counter: Realization of 4-bit up/down counter.</td>
<td>3 hr.</td>
</tr>
<tr>
<td>9</td>
<td>Synchronous Counter: Realization of Mod-N counters.</td>
<td>3 hr.</td>
</tr>
<tr>
<td>10</td>
<td>Ring counter and Johnson Counter. (using FF &amp; 7495).</td>
<td>3 hr.</td>
</tr>
<tr>
<td>11</td>
<td>Realization of counters using IC's (7490, 7492, 7493).</td>
<td>3 hr.</td>
</tr>
<tr>
<td>12</td>
<td>Multiplexers and De-multiplexers using gates and ICs. (74150, 74154)</td>
<td>3 hr.</td>
</tr>
<tr>
<td>13</td>
<td>Realization of combinational circuits using MUX &amp; DEMUX.</td>
<td>3 hr.</td>
</tr>
</tbody>
</table>
13 | Random Sequence generator using LFSR. | 3 hr. |

**PART B (Any 5)**

*The following experiments aim at training the students in digital circuit design with verilog and implementation in small FPGAs. Small, low cost FPGAs, that can be driven by open tools for simulation, synthesis and place and route, such as TinyFPGA or Lattice iCEstick can be used. Open software tools such as yosis (for simulation and synthesis) and arachne (for place and route) may be used. The experiments will lay the foundation for digital design with FPGA with the objective of increased employability.*

<table>
<thead>
<tr>
<th>1</th>
<th>Experiment 1. Realization of Logic Gates and Familiarization of FPGAs</th>
<th>3 hr.</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a)</td>
<td>Familiarization of a small FPGA board and its ports and interface.</td>
<td></td>
</tr>
<tr>
<td>(b)</td>
<td>Create the .pcf files for your FPGA board.</td>
<td></td>
</tr>
<tr>
<td>(c)</td>
<td>Familiarization of the basic syntax of Verilog</td>
<td></td>
</tr>
<tr>
<td>(d)</td>
<td>Development of verilog modules for basic gates, synthesis and implementation in the above FPGA to verify the truth tables.</td>
<td></td>
</tr>
<tr>
<td>(e)</td>
<td>Verify the universality and non associativity of NAND and NOR gates by uploading the corresponding verilog files to the FPGA boards.</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>2</th>
<th>Experiment 2: Adders in Verilog</th>
<th>3 hr.</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a)</td>
<td>Development of verilog modules for half adder in 3 modeling styles (dataflow/structural/behavioural).</td>
<td></td>
</tr>
<tr>
<td>(b)</td>
<td>Development of verilog modules for full adder in structural modeling using half adder.</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>3</th>
<th>Experiment 3: Mux and Demux in Verilog</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>(a)</td>
<td>Development of verilog modules for a 4x1 MUX.</td>
<td></td>
</tr>
<tr>
<td>(b)</td>
<td>Development of verilog modules for a 1x4 DEMUX.</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>4</th>
<th>Experiment 4: Flipflops and counters</th>
<th>3 hr.</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a)</td>
<td>Development of verilog modules for SR, JK and D flipflops.</td>
<td></td>
</tr>
<tr>
<td>(b)</td>
<td>Development of verilog modules for a binary decade/Johnson/Ring counters</td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>5</th>
<th>Experiment 5. Multiplexer and Logic Implementation in FPGA</th>
<th>3 hr.</th>
</tr>
</thead>
<tbody>
<tr>
<td>(a)</td>
<td>Make a gate level design of an 8 : 1 multiplexer, write to FPGA and test its functionality.</td>
<td></td>
</tr>
<tr>
<td>(b)</td>
<td>Use the above module to realize the logic function f (A, B, C) = ( \Sigma m(0, 1, 3, 7) ) and test it.</td>
<td></td>
</tr>
<tr>
<td>(c)</td>
<td>Use the same 8 : 1 multiplexer to realize the logic function f (A, B, C, D) = ( \Sigma m(0, 1, 3, 7, 10, 12) ) by partitioning the truth table properly and test it.</td>
<td></td>
</tr>
<tr>
<td>Experiment</td>
<td>Description</td>
<td>Hours</td>
</tr>
<tr>
<td>------------</td>
<td>-----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------</td>
<td>-------</td>
</tr>
<tr>
<td>6</td>
<td><strong>Experiment 6. Flip-Flops and their Conversion in FPGA</strong></td>
<td>3 hr.</td>
</tr>
<tr>
<td></td>
<td>(a) Make gate level designs of J-K, J-K master-slave, T and D flip-flops, implement and test them on the FPGA board.</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(b) Implement and test the conversions such as T to D, D to T, J-K to T and J-K to D in FPGA</td>
<td></td>
</tr>
<tr>
<td>7</td>
<td><strong>Experiment 7: Asynchronous and Synchronous Counters</strong></td>
<td>3 hr.</td>
</tr>
<tr>
<td></td>
<td>(a) Make a design of a 4-bit up down ripple counter using T-flip-flops in the previous experiment, implement and test them on the FPGA board.</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(b) Make a design of a 4-bit up down synchronous counter using T-flip-flops in the previous experiment, implement and test them on the FPGA board.</td>
<td></td>
</tr>
<tr>
<td>8</td>
<td><strong>Experiment 8: Universal Shift Register in FPGA</strong></td>
<td>3 hr.</td>
</tr>
<tr>
<td></td>
<td>(a) Make a design of a 4-bit universal shift register using D-flip-flops in the previous experiment, implement and test them on the FPGA board.</td>
<td></td>
</tr>
<tr>
<td></td>
<td>(b) Implement ring and Johnson counters with it.</td>
<td></td>
</tr>
<tr>
<td>9</td>
<td><strong>Experiment 9. BCD to Seven Segment Decoder in FPGA</strong></td>
<td>3 hr.</td>
</tr>
<tr>
<td></td>
<td>(a) Make a gate level design of a seven segment decoder, write to FPGA and test its functionality.</td>
<td></td>
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<tr>
<td></td>
<td>(b) Test it with switches and seven segment display. Use output ports for connection to the display.</td>
<td></td>
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<tr>
<td></td>
<td><strong>Total hrs.</strong></td>
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**TEXT/REFERENCE BOOKS:**

<table>
<thead>
<tr>
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COURSE PRE-REQUISITES:

<table>
<thead>
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<th>C.CODE</th>
<th>COURSE NAME</th>
<th>DESCRIPTION</th>
<th>SEM</th>
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<tbody>
<tr>
<td>ECT 203</td>
<td>Logic Circuit Design</td>
<td>To impart an understanding of the basic concepts of Boolean algebra, digital systems which will help them to design and implement different types of practically used sequential circuits using Hardware Description Language.</td>
<td>3rd</td>
</tr>
</tbody>
</table>

COURSE OBJECTIVES:

1. To familiarize students with the Digital Logic Design through the implementation of Logic Circuits using ICs of basic logic gates (ii)
2. To familiarize students with the HDL based Digital Design Flow.

COURSE OUTCOMES:

<table>
<thead>
<tr>
<th>Sl. No</th>
<th>DESCRIPTION</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Design and demonstrate the functioning of various combinational and sequential circuits using ICs</td>
</tr>
<tr>
<td>2</td>
<td>Apply an industry compatible hardware description language to implement digital circuits</td>
</tr>
<tr>
<td>3</td>
<td>Implement digital circuits on FPGA boards and connect external hardware to the boards</td>
</tr>
<tr>
<td>4</td>
<td>Function effectively as an individual and in a team to accomplish the given task</td>
</tr>
</tbody>
</table>

CO-PO-PSO MAPPING

<table>
<thead>
<tr>
<th>CO NO.</th>
<th>PROGRAMME OUTCOMES (POS)</th>
<th>PROGRAMME-SPECIFIC OUTCOMES (PSOS)</th>
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<tbody>
<tr>
<td>1</td>
<td>3 3 3 3 3 3 3 3 3 2 3 3</td>
<td>3 3 3 2 3</td>
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<tr>
<td>2</td>
<td>3 1 1 3 3 3 3 3 3 2 3 3</td>
<td>3 3 3 2 3</td>
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<td>3</td>
<td>3 1 1 3 3 3 3 3 3 2 3 3</td>
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</tr>
<tr>
<td>4</td>
<td>3 3 3 3 3 3 3 3 3 2 3 3</td>
<td>3 3 3 2 3</td>
</tr>
<tr>
<td>ECL 203</td>
<td>3 2 2 3 3 3 3 1 3 2 1 2.5</td>
<td>2.5 2 2.5</td>
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</table>

JUSTIFICATION FOR CO-PO MAPPING

<table>
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<tr>
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<th>LEVEL</th>
<th>JUSTIFICATION</th>
</tr>
</thead>
<tbody>
<tr>
<td>ECL203.1-PO1</td>
<td>3</td>
<td>Design &amp; demonstrate the functioning of various combinational and sequential circuits using ICs to solve complex engineering problems.</td>
</tr>
<tr>
<td>Course Code</td>
<td>PO</td>
<td>Description</td>
</tr>
<tr>
<td>-------------</td>
<td>----</td>
<td>-------------</td>
</tr>
<tr>
<td>ECL203.1-PO2</td>
<td>3</td>
<td>Analysis of complex Engineering problems by demonstrating functioning of various combinational and sequential circuits.</td>
</tr>
<tr>
<td>ECL203.1-PO3</td>
<td>3</td>
<td>Design solutions for various combinational and sequential circuits using ICs.</td>
</tr>
<tr>
<td>ECL203.1-PO9</td>
<td>3</td>
<td>Helps students function effectively as an individual, and as a member or leader in diverse teams, &amp; in multidisciplinary settings.</td>
</tr>
<tr>
<td>ECL203.1-PO12</td>
<td>3</td>
<td>Prepare students to engage in independent and life-long learning in the broadest context of technological change.</td>
</tr>
<tr>
<td>ECL203.2-PO1</td>
<td>3</td>
<td>Application of industry compatible HDL to implement digital circuits helps to solve complex engineering problems.</td>
</tr>
<tr>
<td>ECL203.2-PO2</td>
<td>1</td>
<td>Helps to analyze complex Engineering problems reaching substantiated conclusions using engineering science.</td>
</tr>
<tr>
<td>ECL203.2-PO3</td>
<td>1</td>
<td>Design solutions for complex Engineering problems using industry compatible HDL implementation of digital circuits.</td>
</tr>
<tr>
<td>ECL203.2-PO4</td>
<td>3</td>
<td>Application of HDL in the design of exp. and synthesis of information to provide valid conclusions.</td>
</tr>
<tr>
<td>ECL203.2-PO5</td>
<td>3</td>
<td>Create, select, and apply appropriate techniques in HDL to predict and model complex Engineering activities with an understanding of its limitations.</td>
</tr>
<tr>
<td>ECL203.2-PO9</td>
<td>3</td>
<td>Helps students function effectively as an individual, and as a member or leader in diverse teams, &amp; in multidisciplinary settings.</td>
</tr>
<tr>
<td>ECL203.2-PO12</td>
<td>3</td>
<td>Prepare students to engage in independent and life-long learning in the broadest context of technological change.</td>
</tr>
<tr>
<td>ECL203.3-PO1</td>
<td>3</td>
<td>Design &amp; Implementation of digital circuits on FPGA boards and connect external hardware to the boards helps the students to solve complex engineering problems.</td>
</tr>
<tr>
<td>ECL203.3-PO2</td>
<td>1</td>
<td>Implementation of digital circuits on FPGA board Helps to Analyze complex Engineering problems.</td>
</tr>
<tr>
<td>ECL203.3-PO3</td>
<td>1</td>
<td>Design solutions for complex Engineering problems using FPGA boards.</td>
</tr>
<tr>
<td>ECL203.3-PO4</td>
<td>3</td>
<td>Application of FPGA boards in the design of exp. and synthesis of information to provide valid conclusions.</td>
</tr>
<tr>
<td>ECL203.3-PO5</td>
<td>3</td>
<td>Create, select, and apply appropriate techniques in FPGA to predict and model complex Engineering activities with an understanding of its limitations.</td>
</tr>
<tr>
<td>ECL203.3-PO9</td>
<td>3</td>
<td>Helps students function effectively as an individual, and as a member or leader in diverse teams, &amp; in multidisciplinary settings.</td>
</tr>
<tr>
<td>ECL203.3-PO10</td>
<td>1</td>
<td>Helps students communicate effectively on complex engineering activities with the engineering community and with society.</td>
</tr>
<tr>
<td>ECL203.3-PO12</td>
<td>3</td>
<td>Prepare students to engage in independent and life-long learning in the broadest context of technological change.</td>
</tr>
<tr>
<td>ECL203.4-PO1</td>
<td>3</td>
<td>Engineering knowledge helps students to Function effectively as an individual and in a team to accomplish the given task.</td>
</tr>
<tr>
<td>ECL203.4-PO2</td>
<td>3</td>
<td>Analysis of complex Engineering problems help the students to complete the given task.</td>
</tr>
</tbody>
</table>
Design solutions for given complex Engineering problems as a team or an individual.

Use of modern tools help the students to create, select and apply techniques to complete the given task as an individual or team.

Helps students function effectively as an individual, and as a member or leader in diverse teams, & in multidisciplinary settings.

Prepare students to engage in independent and life-long learning in the broadest context of technological change.

<table>
<thead>
<tr>
<th>MAPPING</th>
<th>LEVEL</th>
<th>JUSTIFICATION</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>ECL203.4-PO3</strong></td>
<td>3</td>
<td>Helps students to demonstrate their skills in designing and testing of digital circuits.</td>
</tr>
<tr>
<td><strong>ECL203.4-PO5</strong></td>
<td>3</td>
<td>Improves ability of students to apply their knowledge and skills in academic or research career with creativity, commitment and social consciousness.</td>
</tr>
<tr>
<td><strong>ECL203.4-PO9</strong></td>
<td>3</td>
<td>Demonstrates a sense of professional ethics to carry out their design of various combinational and sequential circuits using ICs.</td>
</tr>
<tr>
<td><strong>ECL203.4-PO12</strong></td>
<td>3</td>
<td>Helps the students to demonstrate their skills in industry compatible hardware description language to implement digital circuits.</td>
</tr>
<tr>
<td><strong>ECL203.2-PSO1</strong></td>
<td>3</td>
<td>Improves ability of students to apply their knowledge and skills in industrial, academic or research career with creativity, commitment and social consciousness.</td>
</tr>
<tr>
<td><strong>ECL203.2-PSO2</strong></td>
<td>3</td>
<td>Demonstrates a sense of professional ethics to carry out their professional responsibilities in the field of electronics such as implementing digital circuits using industry compatible HDL.</td>
</tr>
<tr>
<td><strong>ECL203.2-PSO3</strong></td>
<td>3</td>
<td>Improves the ability of students to implement digital circuits on FPGA boards and connect external hardware to the boards.</td>
</tr>
<tr>
<td><strong>ECL203.3-PSO1</strong></td>
<td>2</td>
<td>Improves ability of students to apply their knowledge and skills in industrial, academic or research career with creativity, commitment and social consciousness.</td>
</tr>
<tr>
<td><strong>ECL203.3-PSO2</strong></td>
<td>3</td>
<td>Helps students to demonstrate a sense of professional ethics to carry out their professional responsibilities.</td>
</tr>
<tr>
<td><strong>ECL203.3-PSO3</strong></td>
<td>1</td>
<td>Demonstrates their skills in design, implement and test any digital circuits in a team or an individual.</td>
</tr>
<tr>
<td><strong>ECL203.4-PSO1</strong></td>
<td>1</td>
<td>Helps students to demonstrate a sense of professional ethics to carry out their professional responsibilities to complete any task.</td>
</tr>
<tr>
<td><strong>ECL203.4-PSO2</strong></td>
<td>2</td>
<td>Demonstrates their skills in design, implement and test any digital circuits in a team or an individual.</td>
</tr>
</tbody>
</table>
GAPS IN THE SYLLABUS - TO MEET INDUSTRY/PROFESSION REQUIREMENTS:

<table>
<thead>
<tr>
<th>SI No</th>
<th>DESCRIPTION</th>
<th>PROPOSED ACTIONS</th>
<th>PO MAPPING</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Self starting Counters, Code Converters</td>
<td>Assignment</td>
<td>1,2,3,4,5,9</td>
</tr>
</tbody>
</table>

PROPOSED ACTIONS: TOPICS BEYOND SYLLABUS/ASSIGNMENT/INDUSTRY VISIT/GUEST LECTURER/NPTEL ETC

TOPICS BEYOND SYLLABUS/ADVANCED TOPICS/DESIGN:

<table>
<thead>
<tr>
<th>SI No</th>
<th>DESCRIPTION</th>
<th>PO MAPPING</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Advanced design level questions solving skills by lab work to have a wider scope of subject beyond syllabus.</td>
<td>1,2,3,4,5</td>
</tr>
</tbody>
</table>

WEB SOURCE REFERENCES:

2. [http://www.electronics-tutorials.ws/logic/logic_1.html](http://www.electronics-tutorials.ws/logic/logic_1.html)

DELIVERY/INSTRUCTIONAL METHODOLOGIES:

- [CHALK & TALK](#)  [STUD. ASSIGNMENT](#)  [WEB RESOURCES](#)
- [LCD/SMART BOARDS](#)  [STUD. SEMINARS](#)  [ADD-ON COURSES](#)

ASSESSMENT METHODOLOGIES-DIRECT

- [ASSIGNMENTS](#)  [STUD. SEMINARS](#)  [TESTS/MODEL EXAMS](#)  [UNIV. EXAMINATION](#)
- [STUD. LAB PRACTICES](#)  [STUD. VIVA](#)  [MINI/MAJOR PROJECTS](#)  [CERTIFICATIONS](#)
- [ADD-ON COURSES](#)  [OTHERS](#)

ASSESSMENT METHODOLOGIES-INDIRECT

- [ASSESSMENT OF COURSE OUTCOMES (BY FEEDBACK, ONCE)](#)  [STUDENT FEEDBACK ON FACULTY](#)
- [ASSESSMENT OF MINI/MAJOR PROJECTS BY EXT. EXPERTS](#)  [OTHERS](#)
### COURSE PLAN

<table>
<thead>
<tr>
<th>SL.NO</th>
<th>MODULE</th>
<th>PLANNED</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>Realization of functions using basic and universal gates</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>Design and Realization of half/full adder and subtractor using basic gates and universal gates</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>4 bit adder/subtractor and BCD adder using 7483</td>
</tr>
<tr>
<td>4</td>
<td>1</td>
<td>Asynchronous Counter: 3 bit up/down counter</td>
</tr>
<tr>
<td>5</td>
<td>1</td>
<td>Synchronous Counter: Realization of 4-bit up/down counter</td>
</tr>
<tr>
<td>6</td>
<td>1</td>
<td>Adders in Verilog</td>
</tr>
<tr>
<td>7</td>
<td>1</td>
<td>Mux and Demux in Verilog</td>
</tr>
<tr>
<td>8</td>
<td>1</td>
<td>Flipflops and counters</td>
</tr>
<tr>
<td>9</td>
<td>1</td>
<td>Realization of Logic Gates and Familiarization of FPGAs</td>
</tr>
</tbody>
</table>

### LAB CYCLE

The following experiments can be conducted on Tinkercad.

#### CYCLE 1

1. Realization of functions using basic gates
2. Realization of functions using universal gates (SOP and POS forms).
3. Design and Realization of half/full adder and Subtractor using basic gates and universal gates.
4. Implement 4 bit adder/Subtractor using 7483
5. Implement BCD adder using 7483.
6. Realization of combinational circuits using MUX & DEMUX

#### CYCLE 2

1. Asynchronous Counter: 3 bit up/down counter
2. Asynchronous Counter: Realization of Mod N counters
3. Synchronous Counter: Realization of 4-bit up/down counter.
4. Synchronous Counter: Realization of Mod-N counters
5. Ring counter and Johnson Counter. (using FF & 7495).
6. Multiplexers and De-multiplexers using gates and ICs. (74150, 74154)
LIST OF EXPERIMENTS

1. Familiarize the logic gates used in digital integrated circuits and to verify the truth tables.
2. Realize basic logic gates using universal gates – NAND and NOR and to verify the truth tables.
3. Realize SOP and POS functions after K-map reduction.
4. (i) To design and setup half adder, half subtractor, full adder and full subtractor using gates. (ii) To setup half adder and full adder using NAND gates.
5. Design and setup (iii) 4-bit adder circuit using 4-bit binary adder (iii) 4-bit subtractor circuit using 4-bit binary adder (iii) 4-bit adder/subtract circuit using 4-bit binary adder and (iv) 4 bit BCD adder.
6. Set up the following counters
   1. To set up a four bit asynchronous counter as Down counter and Up counter individually
   2. To study its working as a 4-bit down-counter / up-down counter using mode control and to study its working as a 4-bit down-counter / up-down counter.
   3. To study its working as a 4-bit mod-10 asynchronous counter (Decade Counter).
7. Design and setup a Johnson counter and Ring counter using D flip-flops (7474 IC)
8. Design and setup 4 bit synchronous up, down, modulo-10 counter and also setup mode controlled 3 bit up/down counter using 7473.
9. Study and implement the counter ICs 7493
10. Design and set up multiplexer and de-multiplexer using Logic gates
CST 283 PYTHON FOR MACHINE LEARNING (MINOR)
# COURSE INFORMATION SHEET

<table>
<thead>
<tr>
<th>PROGRAMME: MINOR IN COMPUTER SCIENCE &amp; ENGINEERING</th>
<th>DEGREE: B. TECH.</th>
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<tr>
<td>COURSE: Python for Machine Learning</td>
<td>SEMESTER: III</td>
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<tr>
<td>COURSE CODE: CST 283</td>
<td>CREDITS: 4</td>
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<tr>
<td>REGULATION: 2019</td>
<td>COURSE TYPE: MINOR</td>
</tr>
<tr>
<td>COURSE AREA/DOMAIN:</td>
<td>CONTACT HOURS: 3+1 (Tutorial) hours/Week.</td>
</tr>
<tr>
<td>CORRESPONDING LAB COURSE CODE (IF ANY):</td>
<td>LAB COURSE NAME: NIL</td>
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## SYLLABUS:

<table>
<thead>
<tr>
<th>UNIT</th>
<th>DETAILS</th>
<th>HOUR S</th>
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<tbody>
<tr>
<td>II</td>
<td><strong>Building Python Programs:</strong> Control statements - Selection structure (if-else, switch-case), Iteration structure(for, while), Testing the control statements, Lazy evaluation. Functions - Hiding redundancy and complexity, Arguments and return values, Variable scopes and parameter passing, Named arguments, Main function, Working with recursion, Lambda functions. Strings and number systems - String function, Handling numbers in various formats.</td>
<td>8</td>
</tr>
<tr>
<td>III</td>
<td><strong>Data Representation:</strong> Lists - Basic list Operations and functions, List of lists, Slicing, Searching and sorting list, List comprehension. Work with tuples. Sets. Work with dates and times. Dictionaries - Dictionary functions, dictionary literals, adding and removing keys, accessing and replacing values, traversing dictionaries, reverse lookup. Case Study - Data Structure Selection.</td>
<td>9</td>
</tr>
<tr>
<td>IV</td>
<td><strong>Object Oriented Programming:</strong> Design with classes - Objects and Classes, Methods, Instance Variables, Constructor, Accessors and Mutators. Structuring classes with Inheritance and Polymorphism. Abstract Classes. Exceptions - Handle a single exception, handle multiple exceptions.</td>
<td>8</td>
</tr>
<tr>
<td>V</td>
<td><strong>Data Processing:</strong> The os and sys modules. Introduction to file I/O - Reading and writing text files, Manipulating binary files. NumPy - Basics, Creating arrays, Arithmetic, Slicing, Matrix Operations, Random numbers. Plotting and visualization.</td>
<td>10</td>
</tr>
</tbody>
</table>
Matplotlib - Basic plot, Ticks, Labels, and Legends. Working with CSV files.
- Pandas - Reading, Manipulating, and Processing Data.

**TEXT/REFERENCE BOOKS:**

<table>
<thead>
<tr>
<th>T/R</th>
<th>BOOK TITLE/AUTHORS/PUBLICATION</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>Wes McKinney, Python for Data Analysis, 2/e, Shroff / O'Reilly Publishers, 2017</td>
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<td>3</td>
<td>Allen B. Downey, Think Python: How to Think Like a Computer Scientist, 2/e, Schrøff, 2016</td>
</tr>
<tr>
<td>4</td>
<td>Michael Urban and Joel Murach, Python Programming, Shroff/Murach, 2016</td>
</tr>
<tr>
<td>6</td>
<td>Charles Severance. Python for Informatics: Exploring Information</td>
</tr>
<tr>
<td>7</td>
<td><a href="http://swcarpentry.github.io/python-novice-gapminder/">http://swcarpentry.github.io/python-novice-gapminder/</a></td>
</tr>
</tbody>
</table>

**TOTAL HOURS** 45

**COURSE PRE-REQUISITES:** NIL

**COURSE OUTCOMES:**

<table>
<thead>
<tr>
<th>SL. No.</th>
<th>DESCRIPTION</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Write, test and debug Python programs (Cognitive Knowledge level: Apply)</td>
</tr>
<tr>
<td>2</td>
<td>Illustrate uses of conditional (if, if-else, if-elif-else and switch-case) and iterative (while and for) statements in Python programs (Cognitive Knowledge level: Apply)</td>
</tr>
<tr>
<td>3</td>
<td>Develop programs by utilizing the modules Lists, Tuples, Sets and Dictionaries in Python (Cognitive Knowledge level: Apply)</td>
</tr>
<tr>
<td>4</td>
<td>Implement Object Oriented programs with exception handling (Cognitive Knowledge level: Apply)</td>
</tr>
<tr>
<td>5</td>
<td>Write programs in Python to process data stored in files by utilizing the modules Numpy, Matplotlib, and Pandas (Cognitive Knowledge level: Apply)</td>
</tr>
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</table>

**CO-PO-PSO MAPPING:**

<table>
<thead>
<tr>
<th>CO NO.</th>
<th>PROGRAMME OUTCOMES (POS)</th>
<th>PROGRAMME-SPECIFIC OUTCOMES (PSOS)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>PO 1</td>
<td>PO 2</td>
</tr>
<tr>
<td>CO 1</td>
<td>L</td>
<td>H</td>
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</table>
## JUSTIFICATION FOR CO-PO MAPPING

<table>
<thead>
<tr>
<th>MAPPING</th>
<th>LEVEL</th>
<th>JUSTIFICATION</th>
</tr>
</thead>
<tbody>
<tr>
<td>CO1-P01</td>
<td>L</td>
<td>Flowchart and conditional operators used.</td>
</tr>
<tr>
<td>CO1-P02</td>
<td>H</td>
<td>Programs started from the beginning.</td>
</tr>
<tr>
<td>CO1-P05</td>
<td>L</td>
<td>Sort of practical application begins.</td>
</tr>
<tr>
<td>CO1-P011</td>
<td>M</td>
<td>Introduction to python programs</td>
</tr>
<tr>
<td>CO2-P01</td>
<td>L</td>
<td>Functions used in real world applications</td>
</tr>
<tr>
<td>CO2-P02</td>
<td>H</td>
<td>Data types and numerical methods</td>
</tr>
<tr>
<td>CO2-P05</td>
<td>L</td>
<td>Lists and tuples can be useful in real time projects</td>
</tr>
<tr>
<td>CO3-P01</td>
<td>L</td>
<td>Some sort of date and time working in python</td>
</tr>
<tr>
<td>CO3-P02</td>
<td>H</td>
<td>Real time application in projects using searching and sorting algorithms</td>
</tr>
<tr>
<td>CO3-P05</td>
<td>L</td>
<td>Data structure selection has some sort of industrial relevance</td>
</tr>
<tr>
<td>CO4-P01</td>
<td>L</td>
<td>Constructors are some sort of mathematical terms that help students to begin oop.</td>
</tr>
<tr>
<td>CO4-P02</td>
<td>H</td>
<td>Objects and classes are every time used with real time projects</td>
</tr>
<tr>
<td>CO4-P05</td>
<td>M</td>
<td>Inheritance and polymorphism have helped students to survive in industrial live projects</td>
</tr>
<tr>
<td>CO5-P01</td>
<td>H</td>
<td>Matrix operations and random numbers helpful for machine learning algorithms</td>
</tr>
<tr>
<td>CO5-P02</td>
<td>H</td>
<td>The os and sys modules and introduction to text files are very much needed for real time projects</td>
</tr>
<tr>
<td>CO5-P04</td>
<td>M</td>
<td>Plotting and visualization for better understanding of machine learning algorithms</td>
</tr>
<tr>
<td>CO5-P05</td>
<td>H</td>
<td>CSV files for data collection and storage for classification and detection algorithms</td>
</tr>
<tr>
<td>CO5-P011</td>
<td>H</td>
<td>Pandas are helping students to familiarize with other machine learning library.</td>
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</table>

## GAPS IN THE SYLLABUS - TO MEET INDUSTRY/PROFESSION REQUIREMENTS:

<table>
<thead>
<tr>
<th>Sl. No.</th>
<th>DESCRIPTION</th>
<th>PROPOSED ACTIONS</th>
<th>PO MAPPING</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>Familiarization of Tensor Flow</td>
<td>Video links provided</td>
<td>CO5-P02</td>
</tr>
</tbody>
</table>
PROPOSED ACTIONS: TOPICS BEYOND SYLLABUS/ASSIGNMENT/INDUSTRY VISIT/GUEST LECTURER/NPTEL ETC

TOPICS BEYOND SYLLABUS/ADVANCED TOPICS:

<table>
<thead>
<tr>
<th>SL. No.</th>
<th>DESCRIPTION</th>
<th>PO MAPPING</th>
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<tbody>
<tr>
<td>1</td>
<td>Supervised &amp; unsupervised machine learning</td>
<td>CO5-P01</td>
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WEB SOURCE REFERENCES:

<table>
<thead>
<tr>
<th>SL. No.</th>
<th>DESCRIPTION</th>
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<tbody>
<tr>
<td>1</td>
<td><a href="https://www.python.org/">https://www.python.org/</a></td>
</tr>
<tr>
<td>2</td>
<td><a href="https://jupyter.org/try">https://jupyter.org/try</a></td>
</tr>
<tr>
<td>3</td>
<td><a href="https://www.geeksforgeeks.org/">https://www.geeksforgeeks.org/</a></td>
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</table>

Prepared by (Course In-charge)
Approved by
Ms. Binimol Benny
HOD-AEI

COURSE PLAN

<table>
<thead>
<tr>
<th>SL.NO</th>
<th>MODULE</th>
<th>PLANNED</th>
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<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>Getting Started with Python Programming: Running code in the interactive shell Editing, Saving, and Running a script</td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>Using editors: IDLE</td>
</tr>
<tr>
<td>3</td>
<td>1</td>
<td>Jupyter</td>
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<tr>
<td>4</td>
<td>1</td>
<td>The software development process: Case Study.</td>
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<tr>
<td>Chapter</td>
<td>Page</td>
<td>Title</td>
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<tr>
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<tr>
<td>5</td>
<td>1</td>
<td>Basic coding skills: Working with data types, Numeric data types and Character sets, Keywords, Variables and Assignment statement, Operators, Expressions,</td>
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<tr>
<td>6</td>
<td>1</td>
<td>The software development process: Case Study.</td>
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<td>7</td>
<td>1</td>
<td>Working with numeric data, Type conversions, Comments in the program</td>
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<td>8</td>
<td>1</td>
<td>Input, Processing, and Output, Formatting output – How Python works</td>
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<tr>
<td>9</td>
<td>1</td>
<td>How Python works – Detecting and correcting syntax errors</td>
</tr>
<tr>
<td>10</td>
<td>1</td>
<td>Using built in functions and modules: Case – Using math module</td>
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<tr>
<td>11</td>
<td>1</td>
<td>Using built in functions and modules: Case – Using math module (Examples)</td>
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<td>12</td>
<td>2</td>
<td>Control statements: Selection structure (if-else, switch-case),</td>
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<td>13</td>
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<td>Examples based on switch statements</td>
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<td>Iteration structure(for, while), Testing the control statements, Lazy evaluation</td>
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<td>15</td>
<td>2</td>
<td>Functions: Hiding redundancy and complexity, Arguments and return values,</td>
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<td>16</td>
<td>2</td>
<td>Variable scopes and parameter passing</td>
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<td>17</td>
<td>2</td>
<td>Named arguments, Main function,</td>
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<td>18</td>
<td>2</td>
<td>Working with recursion, Lambda functions</td>
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<td>19</td>
<td>2</td>
<td>Strings and number systems: String function</td>
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<tr>
<td>20</td>
<td>2</td>
<td>Handling numbers in various format</td>
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<tr>
<td>21</td>
<td>3</td>
<td>Lists: Basic list Operations and functions, List of lists</td>
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<tr>
<td>22</td>
<td>3</td>
<td>Slicing, Searching and sorting list</td>
</tr>
<tr>
<td>23</td>
<td>3</td>
<td>List comprehension</td>
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<tr>
<td>24</td>
<td>3</td>
<td>Work with tuples, Sets</td>
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<tr>
<td>25</td>
<td>3</td>
<td>Work with dates and times</td>
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<tr>
<td>26</td>
<td>3</td>
<td>Dictionaries: Dictionary functions,</td>
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<tr>
<td>27</td>
<td>3</td>
<td>Dictionary literals, adding and removing keys, accessing &amp; replacing values</td>
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<tr>
<td>28</td>
<td>3</td>
<td>Traversing dictionaries, reverse lookup</td>
</tr>
<tr>
<td>29</td>
<td>3</td>
<td>Case Study: Data Structure Selection</td>
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<tr>
<td>30</td>
<td>4</td>
<td>Design with classes : Objects and Classes, Methods, Instance Variables</td>
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<tr>
<td>31</td>
<td>4</td>
<td>Constructor, Accessors and Mutators</td>
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<td>32</td>
<td>4</td>
<td>Structuring classes with Inheritance</td>
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<td>33</td>
<td>4</td>
<td>Polymorphism</td>
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<td>34</td>
<td>4</td>
<td>Abstract Classes</td>
</tr>
<tr>
<td>35</td>
<td>4</td>
<td>Abstract Classes Continuation</td>
</tr>
<tr>
<td>36</td>
<td>4</td>
<td>Exceptions : Handle a single exception</td>
</tr>
<tr>
<td>37</td>
<td>4</td>
<td>handle multiple exceptions</td>
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<tr>
<td>38</td>
<td>5</td>
<td>The os and sys modules</td>
</tr>
<tr>
<td>39</td>
<td>5</td>
<td>Introduction to file I/O: Reading and writing text files</td>
</tr>
</tbody>
</table>
ASSIGNMENT NO 1

Software development process case study

ASSIGNMENT NO 2

1. Module 2-handling numbers in various formats
2. Case study Data Structure Selection
3. Discuss format specifiers and escape sequences with examples
4. Discuss the relation between tuples, lists, and dictionaries in detail.
5. What are the possible errors in a Python program. Write a Python program to print the value of $2^{(2n)}+n+5$ for $n$ provided by the user.
6. Write a Python code to check whether a given year is a leap year or not [An year is a leap year if it’s divisible by 4 but not divisible by 100 except for those divisible by 400].
7. Write a Python program to check the validity of a password given by the user. The Password should satisfy the following criteria:
   A. Contains at least one letter between a and z
   B. Contains at least one number between 0 and 9
   C. Contains at least one letter between A and Z
   D. Contains at least one special character from $, #, @
   E. Minimum length of password: 6.
TUTORIAL QUESTIONS

TUTORIAL NO-1

Q1) PROGRAM USING SWITCH CASE STATEMENT FOR PRINTING DAYS IN A WEEK
Q2) PROGRAM USING SWITCH CASE STATEMENT IN PYTHON TO DO THE FUNCTIONS OF A CALCULATOR.

TUTORIAL NO-2

1. PROGRAM TO DO SUM OF DIGITS OF A NUMBER USING FUNCTIONS
2, PROGRAM TO FACTORIAL OF A NUMBER USING FUNCTIONS

TUTORIAL NO 3

Explain the types of inheritance with detailed examples. Draw diagram for all the types

TUTORIAL NO 4

Write the program and output and code the program using numpy in python
a) matrix multiplication using two 3*3 matrices matrix A and Matrix B
b) To print the rows in the Matrix A
c) To read the last element from each row of Matrix B